Answer on Question #65599 - Math - Linear Algebra

Question:

1) Which of the following statements are true and which are false? Justify your answer with a short proof or a counterexample.

   i) The relation $\sim$ defined by $\boldsymbol{R}$ by $x \sim y$ if $x \geq y$ is an equivalence relation.
   ii) If S_1 and S_2 are finite non-empty subsets of a vector space V such that $[S_1] = [S_2]$, then S_1 and S_2 have the same number of elements.
   iii) For any square matrix A, $\rho(A) = det(A)$.
   iv) The determinant of any unitary matrix is 1.
   v) If the characteristic polynomials of two matrices are equal, their minimal polynomials are also equal.
   vi) If the determinant of a matrix is 0, the matrix is not diagonalisable.
   vii) Any set of mutually orthogonal vectors is linearly independent.
   viii) Any two real quadratic forms of the same rank are equivalent over $\boldsymbol{R}$.
   ix) There is no system of linear equations over $\boldsymbol{R}$ that has exactly two solutions.
   x) If a square matrix A satisfies the equation $A^2 = A$, then 0 and 1 are the eigenvalues of A.

Answers

i) **No.** The relation $\sim$ is not symmetric. For example, $5 \geq 4$, but $4 \geq 5$ is not valid.

ii) **No.** If [S] is a span of S, then the answer is negative. The subsets $S_1 = \{(1{,}0),(2{,}0)\}$ and $S_2 = \{(1{,}0)\}$ have the same spans.

iii) **No.** $\rho(A) = \max\{|\lambda_1|, |\lambda_2|, \cdots |\lambda_n|\}$ is a spectral radius of the matrix A. In general $\rho(A) \neq \det A$, since $\det A = \lambda_1 \cdot \lambda_2 \cdots \lambda_n$.

iv) No. The matrix $U = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$ is uniform. Indeed, its conjugate transpose $U^* = \begin{bmatrix} 1 & 0 \\ 0 & -i \end{bmatrix}$ and $U^*U = UU^* = I$. But $\det U = i \neq 1$.

v) No. The counterexample is the following:

	Characteristic polynomial	Minimal polynomial
$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\det(A_1 - \lambda I) = (\lambda - 1)^2$	$\lambda - I$
$A_2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$	$\det(A_2 - \lambda I) = (\lambda - 1)^2$	$(\lambda - I)^2$

vi) No. For the matrix $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $\det A = 0$, but A is diagonalisable.

vii) No. The statement is not generally true, since the system can contain zero vectors. But the statement is true, if all vectors are nonzero.

viii) **No.** Let us consider such two real quadratic forms as $q_1(x) = x_1^2 + x_2^2$ and $q_2(x) = x_1^2 - x_2^2$. Those two quadratic forms have the same rank $r = 2$, but they are not equivalent, because the signatures of the forms are different.

ix) Yes. If the system of linear equations over $\boldsymbol{R}$ has some two solutions, then it has infinitely many solutions. If x_1 and x_2 are two different solutions, then the vector $x_1 + C\,(x_2 - x_1)$ is a solution as well for any real constant C.

x) No. Let us consider the matrix $A = 0$, where 0 denotes the null-matrix. It satisfies the equation $A^2 = A$, but 1 is not an eigenvalue of A.