Answer on Question #65598 - Math - Statistics and Probability

Question: For the random variable X with the following probability density function

$$f(x) = \begin{cases} 2e^{-2x}, & 0 \geq x; \\ 0, & 0 < x \end{cases}$$

find

i) $P\{|X-\mu| > 1\}$;

ii) Use Chebyshev's inequality to obtain an upper bound on $P\{|X-\mu| > 1\}$ and compare with the result in (i).

Solution: First of all the function f is not probability density function for any random variable X. Indeed,

$$\int_{-\infty}^{+\infty} f(x)dx = \int_{-\infty}^{0} 2e^{-2x}dx = \int_{-\infty}^{0} e^{-2x}d(2x) = -e^{-2x}\Big|_{-\infty}^{0} = +\infty \neq 1.$$

Therefore, the correct probability density function (pdf) should be define as follows:

$$f(x) = \begin{cases} 2e^{-2x}, & x \geq 0; \\ 0, & x < 0 \end{cases}$$

or

$$f(x) = \begin{cases} 2e^{2x}, & 0 \geq x; \\ 0, & 0 < x. \end{cases}$$

I will consider the first function

$$f(x) = \begin{cases} 2e^{-2x}, & x \geq 0; \\ 0, & x < 0, \end{cases}$$

which is a probability density function of the exponential distribution.

The expectation for X is equal to

$$\mu = \int_{-\infty}^{+\infty} xf(x)dx$$

$$= \int_{0}^{+\infty} x \cdot 2e^{-2x}dx = -\int_{0}^{+\infty} xd(e^{-2x}) = -\left(xe^{-2x}\Big|_0^{+\infty} - \int_0^{+\infty} e^{-2x}dx\right)$$

$$= -\left(\lim_{x\to+\infty}\frac{x}{e^{2x}} - 0 + \frac{1}{2}e^{-2x}\Big|_0^{+\infty}\right) = 0 - \frac{1}{2}\left(\lim_{x\to+\infty} e^{-2x} - 1\right) = -\frac{1}{2}(0-1) = \frac{1}{2}.$$

i) $P\{|X-\mu| > 1\} = P(\{X > \mu+1\} \cup \{X < \mu - 1\}) = P\left(\{X > \frac{3}{2}\} \cup \{X < -\frac{1}{2}\}\right) = \int_{3/2}^{+\infty} 2e^{-2x}dx + \int_{-\infty}^{-1/2} 0\,dx = \int_{3/2}^{+\infty} e^{-2x}d(2x) = -e^{-2x}\Big|_{\frac{3}{2}}^{+\infty} = e^{-2\cdot\frac{3}{2}} = e^{-3}.$

ii) By Chebyshev's inequality we have $P\{|X-\mu| > 1\} \leq \frac{Var[X]}{1^2} = Var[X]$.

Let us calculate the variance $Var[X] = E[X^2] - (E[X])^2$.

$E[X^2] = \int_{-\infty}^{+\infty} x^2 \cdot f(x)dx = \int_0^{+\infty} x^2 \cdot 2e^{-2x}dx = -\int_0^{+\infty} x^2 d(e^{-2x}) = -\left(x^2e^{-2x}\Big|_0^{+\infty} - \int_0^{+\infty} e^{-2x}\cdot 2xdx\right) = \int_0^{+\infty} e^{-2x}\cdot 2xdx = \frac{1}{2}$

,

$$Var[X] = \frac{1}{2} - \left(\frac{1}{2}\right)^2 = \frac{1}{4}.$$

Hence, $P\{|X-\mu| > 1\} \leq 0.25$ by Chebyshev's inequality.

Finally, we have that the exact value of probability $P\{|X-\mu| > 1\}$ equals $e^{-3} \approx 0.049787$ and the approximation obtained by Chebyshev's inequality is equal to 0.25. This gives

$\frac{|0.049787-0.25|}{0.049787} \cdot 100\% \approx 402\%$ of percent error. In other words, the upper bound is 5 times greater than the exact value of probability.

Answer: $P\{|X-\mu|>1\} = e^{-3} \approx 0.049787$, $P\{|X-\mu|>1\} \leq 0.25$ by Chebyshev's inequality.

For Chebyshev's inequality see for example
W. Feller, "An introduction to probability theory and its applications", **1**, Wiley (1957-1971), P.233,
and for pdf and its properties
W. Feller, "An introduction to probability theory and its applications", **2**, Wiley (1971), P.3-5

https://www.encyclopediaofmath.org/index.php/Feller,_%22An_introduction_to_probability_theory_and_its_applications%22