Answer on Question #65386 - Math - Statistics and Probability

Question: Let Y, X have joint pdf

$$f_{XY}(x,y) = \begin{cases} 4, & 0 \le x \le 1, 0 \le y \le 1; \\ 0, & otherwise \end{cases}$$

Find $f_X(x), f_Y(y), f_{X/Y}(x/y), f_{Y/X}(y/x)$, also find $E(X/|Y = y)$and $E(Y/|X = x)$.

Solution: First of all the given function is not joint pdf for any random variables Y, X. Indeed,

$$\int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty} f_{XY}(x,y)dy\right)dx = \int_0^1\left(\int_0^1 4dy\right)dx = 4\cdot 1\cdot 1 = 4 \ne 1.$$

Therefore, in order to obtain a joint pdf we have to change the constant 4 or the domain $0 \le x \le 1, 0 \le y \le 1$ in the description of $f_{XY}(x,y)$.

Let us consider such joint pdf

$$f_{XY}(x,y) = \begin{cases} a, & 0 \le x \le \frac{1}{\sqrt{a}}, 0 \le y \le \frac{1}{\sqrt{a}}; \\ 0, & otherwise, \end{cases}$$

where a is an arbitrary positive real number.

The marginal density functions are

$f_X(x) = \int_{-\infty}^{+\infty} f_{XY}(x,y)\,dy = \int_0^{1/\sqrt{a}} a\,dy = a\,y|_0^{1/\sqrt{a}} = \sqrt{a}$, when $0 \le x \le \frac{1}{\sqrt{a}}$,

and

$f_Y(y) = \int_{-\infty}^{+\infty} f_{XY}(x,y)\,dx = \int_0^{1/\sqrt{a}} a\,dx = a\,x|_0^{1/\sqrt{a}} = \sqrt{a}$, when $0 \le y \le \frac{1}{\sqrt{a}}$.

For $0 \le y \le \frac{1}{\sqrt{a}}$

$f_{X/Y}(x/y) = \frac{f_{XY}(x,y)}{f_Y(y)} = \frac{a}{\sqrt{a}} = \sqrt{a}$, when $0 \le x \le \frac{1}{\sqrt{a}}$;

otherwise $f_{X/Y}(x/y) = 0$.

For $0 \le x \le \frac{1}{\sqrt{a}}$

$f_{Y/X}(y/x) = \frac{f_{XY}(x,y)}{f_X(x)} = \frac{a}{\sqrt{a}} = \sqrt{a}$, when $0 \le y \le \frac{1}{\sqrt{a}}$;

otherwise $f_{Y/X}(y/x) = 0$.

Let us now calculate corresponding conditional expectations

$E(X|Y = y) = \int_{-\infty}^{+\infty} x\; f_{X/Y}(x/y)dx = \int_0^{1/\sqrt{a}} \sqrt{a}\cdot x\,dx = \sqrt{a}\cdot \frac{x^2}{2}\Big|_0^{1/\sqrt{a}} = \frac{1}{2\sqrt{a}}$, when $0 \le y \le \frac{1}{\sqrt{a}}$.

$E(Y|X = x) = \int_{-\infty}^{+\infty} y\; f_{Y/X}(y/x)dy = \int_0^{1/\sqrt{a}} \sqrt{a}\cdot y\,dy = \sqrt{a}\cdot \frac{y^2}{2}\Big|_0^{1/\sqrt{a}} = \frac{1}{2\sqrt{a}}$, when $0 \le x \le \frac{1}{\sqrt{a}}$.

Answer: For a joint pdf

$$f_{XY}(x,y) = \begin{cases} a, & 0 \le x \le \frac{1}{\sqrt{a}}, 0 \le y \le \frac{1}{\sqrt{a}}; \\ 0, & otherwise, \end{cases}$$

$$f_X(x) = \begin{cases} \sqrt{a}, & 0 \le x \le \frac{1}{\sqrt{a}}, \\ 0, & otherwise; \end{cases} \quad f_Y(y) = \begin{cases} \sqrt{a}, & 0 \le y \le \frac{1}{\sqrt{a}}, \\ 0, & otherwise; \end{cases}$$

For $0 \leq y \leq \frac{1}{\sqrt{a}}$ $f_{X/Y}(x/y) = \begin{cases} \sqrt{a}, & 0 \leq x \leq \frac{1}{\sqrt{a}}, \\ 0, otherwise; \end{cases}$ and $E(X|Y = y) = \frac{1}{2\sqrt{a}}$;

For $0 \leq x \leq \frac{1}{\sqrt{a}}$ $f_{Y/X}(y/x) = \begin{cases} \sqrt{a}, & 0 \leq y \leq \frac{1}{\sqrt{a}}, \\ 0, otherwise; \end{cases}$ and $E(Y|X = x) = \frac{1}{2\sqrt{a}}$.

For marginal density function, conditional density function and conditional expectation see for example W. Feller, "An introduction to probability theory and its applications", **2**, Wiley (1971), P.66-67 and 71-72.
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