Answer on Question 65076 - Math - Differential Equations

Question: Find the equation of the integral surface of the differential equation $(x^2 - yz)p + (y^2 - zx)q = z^2 - xy$ which passes through the line $x = 1$, $y = 0$.

Solution: Consider a quasilinear equation

$$a(x, y, z)p + b(x, y, z)q = c(x, y, z).$$

By Lagrange's method the auxiliary equations are as following:

$$\frac{dx}{a(x, y, z)} = \frac{dy}{b(x, y, z)} = \frac{dz}{c(x, y, z)}.$$

So, for the given quasilinear equation we come to the system in the symmetric form

$$\frac{dx}{x^2 - yz} = \frac{dy}{y^2 - zx} = \frac{dz}{z^2 - xy}. \tag{1}$$

One of a way to solve the system in symmetric form is to use the equal fractions property

$$\frac{a_1}{b_1} = \frac{a_2}{b_2} = \cdots = \frac{a_n}{b_n} = \frac{\lambda_1 a_1 + \lambda_2 a_2 + \cdots + \lambda_n a_n}{\lambda_1 b_1 + \lambda_2 b_2 + \cdots + \lambda_n b_n}.$$

In our case we have

$$\frac{dx - dy}{x^2 - yz - y^2 + zx} = \frac{dy - dz}{y^2 - zx - z^2 + xy},$$

$$\frac{d(x - y)}{x^2 - y^2 + z(x - y)} = \frac{d(y - z)}{y^2 - z^2 + x(y - z)},$$

$$\frac{d(x - y)}{(x - y)(x + y + z)} = \frac{d(y - z)}{(y - z)(x + y + z)},$$

$$\frac{d(x - y)}{x - y} = \frac{d(y - z)}{y - z}.$$

Integrating of the last equality yields

$$\int \frac{d(x - y)}{x - y} = \int \frac{d(y - z)}{y - z}$$

$$\Rightarrow \quad \ln|x - y| = \ln|y - z| + \ln|C_1| \quad \Rightarrow \quad \frac{x - y}{y - z} = C_1.$$

Since the system (1) is invariant under the cyclic interchange of variables $x \mapsto y \mapsto z \mapsto x$, there exist two integrals

$$\frac{x - y}{y - z} = C_1, \qquad \frac{y - z}{z - x} = C_2.$$

Therefore any integral surface of the differential equation $(x^2 - yz)p + (y^2 - zx)q = z^2 - xy$ is described by the equation

$$F\left(\frac{x-y}{y-z}, \frac{y-z}{z-x}\right) = 0,$$

where F is a smooth function. If we substitute the conditions $x = 1$ and $y = 0$, then

$$F\left(-\frac{1}{z}, \frac{-z}{z-1}\right) = 0 \text{ or } F\left(\tau, -\frac{1}{\tau+1}\right) = 0.$$

We see that the last relation does not define the function F uniquely. Hence, the problem admits many solutions.

Answer: $F\left(\frac{x-y}{y-z}, \frac{y-z}{z-x}\right) = 0$, where $F = F(\xi, \eta)$ is a C^1-function such that $F\left(\tau, -\frac{1}{\tau+1}\right) = 0$.