Answer on Question #64862 – Math – Real Analysis

Question

Verify the second mean value theorem of integrability for the functions f and g defined on $[1,2]$ by $f(x) = 3x$ and $g(x) = 5x$.

Solution

Let us state the second mean value theorem.

If $f: [a,b] \to \mathbb{R}$ is a monotonic function and $g: [a,b] \to \mathbb{R}$ is an integrable function, then there exists a number x in (a,b) such that

$$\int_a^b f(t)g(t)dt = f(a^+)\int_a^x g(t)dt + f(b^-)\int_x^b g(t)dt.$$

Function $f(x) = 3x$ increases on $[1,2]$;

$\int_1^2 g(t)dt = \int_1^2 5tdt = \frac{5t^2}{2}\Big|_1^2 = \frac{15}{2} < \infty$, so $g(x) = 5x$ is integrable on $[1,2]$.

Then

$$\int_1^2 15t^2dt = f(1^+)\int_1^x 5tdt + f(2^-)\int_x^2 5tdt;$$

$$\int_1^2 15t^2dt = 15\int_1^x tdt + 30\int_x^2 tdt;$$

$$5t^3\big|_{t=1}^2 = \frac{15t^2}{2}\Big|_{t=1}^x + 15t^2\big|_{t=x}^2;$$

$$35 = \frac{15x^2}{2} - \frac{15}{2} + 60 - 15x^2.$$

We must solve this equation on $(1,2)$.

$$-\frac{15x^2}{2} = -\frac{35}{2} \Rightarrow x^2 = \frac{7}{3} \Rightarrow x = \sqrt{\frac{7}{3}} \in (1,2).$$

So we found $x = \sqrt{\frac{7}{3}} \in (1,2)$ such that

$\int_1^2 f(t)g(t)dt = f(1^+)\int_1^x g(t)dt + f(2^-)\int_x^2 g(t)dt$. The theorem is verified.

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