# Answer on Question #64858 – Math – Linear Algebra

## Question

Let $T: P^2 \to P^1$ be defined by

$$T(a+bx+cx^2)=b+c+(a-c)x.$$

Check that $T$ is a linear transformation.

Find the matrix of the transformation with respect to the ordered basis

$$B_1=\{x^2, x^2+x,\ x^2+x+1\}$$

and

$$B_2=\{1,x\}.$$

Find the kernel of $T$.

## Solution

A linear transformation **[1]** between two vector spaces **[2]** $V$ and $W$ is a map **[3]** $T: V \to W$ such that the following properties hold:

(1) $T(v_1+v_2)=T(v_1)+T(v_2)$ for any vectors $v_1, v_2 \in V$;

(2) $T(\alpha \cdot v)=\alpha T(v)$ for any scalar $\alpha$ and any $v \in V$.

Let's check the properties in our case:

**(1)** Let $p=k_0+k_1x+k_2x^2$ and $q=l_0+l_1x+l_2x^2$ be polynomials in $P^2$.

Note that

$$p+q=k_0+k_1x+k_2x^2+l_0+l_1x+l_2x^2=(k_0+l_0)+(k_1+l_1)x+(k_2+l_2)x^2$$

and by definition of $T$ we get:

$$T(p+q)=T((k_0+l_0)+(k_1+l_1)x+(k_2+l_2)x^2)=(k_1+l_1)+(k_2+l_2)+((k_0+l_0)-(k_2+l_2))x=$$
$$=k_1+l_1+k_2+l_2+(k_0+l_0-k_2-l_2)x$$

and

$$T(p)+T(q)=T(k_0+k_1x+k_2x^2)+T(l_0+l_1x+l_2x^2)=k_1+k_2+(k_0-k_2)x+l_1+l_2+(l_0-l_2)x=$$
$$=k_1+l_1+k_2+l_2+(k_0+l_0-k_2-l_2)x$$

Thus, we can see that

$$T(p+q)=T(p)+T(q),$$

so the property (1) holds.

**(2)** Let $p=k_0+k_1x+k_2x^2$ and let $\alpha$ be a scalar.

$$T(\alpha p)=T(\alpha k_0+\alpha k_1x+\alpha k_2x^2)=\alpha k_1+\alpha k_2+(\alpha k_0-\alpha k_2)x=\alpha(k_1+k_2)+\alpha(k_0-k_2)x,$$

while

$$\alpha T(p)=\alpha((k_1+k_2)+(k_0-k_2)x)=\alpha(k_1+k_2)+\alpha(k_0-k_2)x.$$

Therefore,

$$T(\alpha p)=\alpha T(p),$$

so the property (2) holds as well.

Thus, $T$ is a linear transformation.

To find the matrix of the transformation with respect to the ordered basis we need to apply the transformation to the basis and the result is the column of the transformation matrix.

$$T(x^2)=\left[here\ a=0,\ b=0,\ c=1,\ and\ a-c=-1,\ b+c=1\right]=1-x;$$

$$T(x+x^2)=\left[here\ a=0,\ b=1,\ c=1,\ and\ a-c=-1,\ b+c=2\right]=2-x;$$

$$T(1+x+x^2)=\left[here\ a=1,\ b=1,\ c=1,\ and\ a-c=0,\ b+c=2\right]=2;$$

Next, we find the coordinates for each of the above polynomials in the second basis:

$$1-x=1\cdot 1+(-1)\cdot x=(1,-1);$$

$$2-x=2\cdot 1+(-1)\cdot x=(2,-1);$$

$$2=2\cdot 1+0\cdot x=(2,0);$$

Thus, the matrix representation of $T$ with respect to the given basies is

$$T[B_1,B_2]=\begin{pmatrix} 1 & 2 & 2 \\ -1 & -1 & 0 \end{pmatrix}$$

For transormation $T:P^2\to P^1$ the kernel **[4]** (also called the null space **[5]**) is defined by

$$Ker\,(T)=\{p\in P^2 : T(p)=0\},$$

so the kernel gives the elements from the original set $P^2$ that are mapped to zero by the transformation.

Let $p=k_0+k_1x+k_2x^2$ is arbitrarily polynomial from $P^2$. Note that

$$T(k_0+k_1x+k_2x^2)=k_1+k_2+(k_0-k_2)x,$$

and solve the equation

$$k_1+k_2+(k_0-k_2)x=0.$$

Two polynomials are equal if and only if the coefficients of the corresponding powers are equal, hence we get the system of two equations:

$$\begin{cases} k_0-k_2=0, & (1) \\ k_1+k_2=0. & (2) \end{cases}$$

It follows from (1) that

$$k_2=k_0$$

and substituting into (2) one gets

$$k_1=-k_0.$$

Thus, the polynomial $p$ belongs to the kernel of the transformation $T$ if $p$ has the form

$$p=k_0-k_0x+k_0x^2$$

and

$$Ker\,(T)=\{p\in P^2 :\ p=k_0-k_0x+k_0x^2\}.$$

**Answer:**

$T$ is a linear transformation;

$T[B_1,B_2]=\begin{pmatrix} 1 & 2 & 2 \\ -1 & -1 & 0 \end{pmatrix};$

$Ker\,(T)=\{p\in P^2 :\ p=k_0-k_0x+k_0x^2\}.$

**References:**


1. Beezer, R. (2015) A first Course in Linear Algebra. Subsection LT: Linear Transformations. Retrieved from http://linear.ups.edu/html/section-LT.html.
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