Answer on Question #64856 – Math – Linear Algebra

Question

Find the orthogonal canonical reduction of the quadratic form

$$-x^2 + y^2 + z^2 + 2xy - 2xz + 2yz.$$

Also, find its principal axes.

Solution

Denote by f this quadratic form. Then $f(R) = R^T AR$, where

$$A = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ -1 & 1 & 1 \end{pmatrix}$$

is a 3x3 matrix,

$$R = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

is a column (3x1 matrix).
So, it is required to find a orthogonal matrix Q such that

$$Q^T Q = E$$

and

$$f(R) = d_1 x'^2 + d_2 y'^2 + d_3 z'^2,$$

where

$$R = QR',$$

$$R' = \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix},$$

i.e., $f(R) = R'^T DR'$, where D is a diagonal 3x3 matrix,

$$D = \begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{pmatrix},$$

E is a unit matrix.
Then

$$f(R) = R'^T Q^T AQR'.$$

Therefore $D = Q^T AQ$ and $AQ = QD$.
Then $\sum_j^3 A_{i\,j} Q_{j\,k} = d_k Q_{i\,k}$ (the first index is the number of a row and the second index is the number of a column). So the columns of Q are eigenvectors of A and $d_1,\ d_2,\ d_3$ are eigenvalues of A.
Eigenvalues of A are roots of the polynomial

$$\det(\mathrm{A} - d\ \mathrm{E}) = \det\begin{pmatrix} 1-d & 1 & -1 \\ 1 & 1-d & 1 \\ -1 & 1 & 1-d \end{pmatrix} = -d^3 + 3d^2 - 4 = -(d+1)(d-2)^2.$$

This polynomial has a single root -1 and double root 2.
The coordinates of the eigenvector of A associated with eigenvalue -1 are a solution of a linear system $(\mathrm{A} + \mathrm{E})\mathrm{U} = 0$ with respect to U. This system can be written as

$$\begin{cases} 2u_1 + u_2 - u_3 = 0 \\ u_1 + 2u_2 + u_3 = 0 \\ -u_1 + u_2 + 2u_3 = 0 \end{cases}$$

Hence $u_1 = u_3$, $u_2 = -u_3$. Q is orthogonal, it follows that $u_1^2 + u_2^2 + u_3^2 = 1 = 3u_3^2$. Therefore $u_1 = \frac{1}{\sqrt{3}}, u_2 = -\frac{1}{\sqrt{3}}, u_3 = \frac{1}{\sqrt{3}}$.

The coordinates of the eigenvectors of A with eigenvalues 2 are a solution of a linear system $(\mathrm{A} - 2\mathrm{E})\mathrm{U} = 0$ with respect to U:

$$\begin{cases} -u_1 + u_2 - u_3 = 0 \\ u_1 - u_2 + u_3 = 0 \\ -u_1 + u_2 - u_3 = 0 \end{cases}$$

The rank of this system is equal to 1 and there are two linearly independent solutions of this system. It follows from $u_1 = -u_3/2, u_2 = u_3/2$. $u_1^2 + u_2^2 + u_3^2 = 1 = \frac{3}{2}u_3^2$ that the first solution is

$$u_1 = -\frac{1}{\sqrt{6}}, u_2 = \frac{1}{\sqrt{6}}, u_3 = \frac{\sqrt{2}}{\sqrt{3}}.$$

The solution that is orthogonal to it is a solution of the system

$$\begin{cases} -u_1 + u_2 - u_3 = 0, \\ -\frac{1}{\sqrt{6}}u_1 + \frac{1}{\sqrt{6}}u_2 + \frac{\sqrt{2}}{\sqrt{3}}u_3 = 0, \end{cases}$$

that is,

$$\begin{cases} -u_1 + u_2 - u_3 = 0, \\ -u_1 + u_2 + 2u_3 = 0. \end{cases}$$

Then

$$u_3 = 0, u_1 = u_2, u_1^2 + u_2^2 + u_3^2 = 1 = 2u_1^2, \text{ hence } u_1 = u_2 = \frac{1}{\sqrt{2}}, u_3 = 0.$$

Therefore

$$Q = \begin{pmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{\sqrt{2}}{\sqrt{3}} & 0 \end{pmatrix},$$

$$x = \frac{1}{\sqrt{3}}x' - \frac{1}{\sqrt{6}}y' + \frac{1}{\sqrt{2}}z'$$
$$y = -\frac{1}{\sqrt{3}}x' + \frac{1}{\sqrt{6}}y' + \frac{1}{\sqrt{2}}z',$$
$$z = \frac{1}{\sqrt{3}}x' + \frac{\sqrt{2}}{\sqrt{3}}y',$$

$$x' = \frac{1}{\sqrt{3}}x - \frac{1}{\sqrt{3}}y + \frac{1}{\sqrt{3}}z,$$
$$y' = -\frac{1}{\sqrt{6}}x + \frac{1}{\sqrt{6}}y + \frac{\sqrt{2}}{\sqrt{3}}z,$$
$$z' = \frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}y.$$

Principal axes are such columns e_1, e_2, e_3 that are a basis and the coordinates of R in this basis are x', y', z':

$$R = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = x'e_1 + y'e_2 + z'e_3$$

for any R.

It means that $R = (e_1, e_2, e_3)R'$, where (e_1, e_2, e_3) is a matrix composed by columns e_1, e_2, e_3. Therefore $Q = (e_1, e_2, e_3)$ and the principal axes are the columns of Q.

Answer:

The orthogonal canonical reduction is

$$x = \frac{1}{\sqrt{3}}x' - \frac{1}{\sqrt{6}}y' + \frac{1}{\sqrt{2}}z'$$
$$y = -\frac{1}{\sqrt{3}}x' + \frac{1}{\sqrt{6}}y' + \frac{1}{\sqrt{2}}z',$$
$$z = \frac{1}{\sqrt{3}}x' + \frac{\sqrt{2}}{\sqrt{3}}y'$$

$$x' = \frac{1}{\sqrt{3}}x - \frac{1}{\sqrt{3}}y + \frac{1}{\sqrt{3}}z$$
$$y' = -\frac{1}{\sqrt{6}}x + \frac{1}{\sqrt{6}}y + \frac{\sqrt{2}}{\sqrt{3}}z.$$
$$z' = \frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}y$$

Then the reduced form is

$$-x'^2 + 2y'^2 + 2z'^2.$$

The principal axes are

$$\begin{pmatrix} \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix}, \begin{pmatrix} -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \frac{\sqrt{2}}{\sqrt{3}} \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}.$$