Answer on Question #64774 – Math – Differential Equations

Question

Solve (D4+8D2−9)y=9x3+5cos2x

Solution

Rewrite the equation $(D^4 + 8D^2 - 9)y = 9x^3 + 5\cos 2x$ as

$$y^{(4)} + 8y'' - 9y = 9x^3 + 5\cos 2x.$$

The general solution will be the sum of the complementary and a particular solutions. Find the complementary solution by solving

$$y^{(4)} + 8y'' - 9y = 0.$$

Assume a solution will be $e^{\lambda x}$, where λ is a constant.

Substituting $y = e^{\lambda x}$ into the differential equation

$$(e^{\lambda x})^{(4)} + 8(e^{\lambda x})'' - 9e^{\lambda x} = 0,$$

$$\lambda^4 e^{\lambda x} + 8\lambda^2 e^{\lambda x} - 9e^{\lambda x} = 0,$$

$$e^{\lambda x}(\lambda^4 + 8\lambda^2 - 9) = 0,$$

$$e^{\lambda x} \neq 0, \ \lambda^4 + 8\lambda^2 - 9 = 0,$$

$$(\lambda^2 + 9)(\lambda^2 - 1) = 0,$$

$$(\lambda + 3i)(\lambda - 3i)(\lambda - 1)(\lambda + 1) = 0.$$

Solving for λ:

$$\lambda = 1 \text{ or } \lambda = -1 \text{ or } \lambda = 3i \text{ or } \lambda = -3i.$$

The complementary solution is

$$y_c = C_1 e^x + C_2 e^{-x} + C_3 \cos 3x + C_4 \sin 3x.$$

Determine a particular solution of $y^{(4)} + 8y'' - 9y = 9x^3 + 5\cos 2x$ via the method of undetermined coefficients.

A particular solution will be the sum of the particular solutions to $y^{(4)} + 8y'' - 9y = 9x^3$ and $y^{(4)} + 8y'' - 9y = 5\cos 2x$.

A particular solution to $y^{(4)} + 8y'' - 9y = 9x^3$ is given by

$$y_1 = a_1 + a_2 x + a_3 x^2 + a_4 x^3.$$

A particular solution to $y^{(4)} + 8y'' - 9y = 5\cos 2x$ is given by

$$y_2 = a_5 \cos 2x + a_6 \sin 2x.$$

Adding y_1 and y_2 one obtains y_p:

$$y_p = y_1 + y_2 = a_1 + a_2 x + a_3 x^2 + a_4 x^3 + a_5 \cos 2x + a_6 \sin 2x.$$

To find the unknown coefficients a_1, a_2, a_3, a_4, a_5 and a_6 first we need to compute

$$(y_p)^{(4)} = \left(a_1 + a_2 x + a_3 x^2 + a_4 x^3\right)^{(4)} + \left(a_5 \cos 2x + a_6 \sin 2x\right)^{(4)} =$$

$$= a_5 \cdot 2^4 \cos\left(2x + \frac{\pi}{2} \cdot 4\right) + a_6 \cdot 2^4 \sin\left(2x + \frac{\pi}{2} \cdot 4\right) = 16a_5 \cos 2x + 16a_6 \sin 2x$$

$$(y_p)'' = \left(a_1 + a_2 x + a_3 x^2 + a_4 x^3\right)'' + \left(a_5 \cos 2x + a_6 \sin 2x\right)'' = 2a_3 + 6a_4 x - 4a_5 \cos 2x - 4a_6 \sin 2x$$

Substituting the particular solution y_p and $(y_p)^{(4)}$, $(y_p)''$ into the differential equation

$$(y_p)^{(4)} + 8(y_p)'' - 9y = 9x^3 + 5\cos 2x,$$

one gets

$$16a_5\cos 2x + 16a_6\sin 2x + 8(2a_3 + 6a_4x - 4a_5\cos 2x - 4a_6\sin 2x) -$$
$$-9(a_1 + a_2x + a_3x^2 + \ a_4x^3 + a_5\cos 2x + a_6\sin 2x) = 9x^3 + 5\cos 2x.$$

Simplifying

$$(-9a_1 + 16a_3) + (-9a_2 + 48a_4) - 9a_3x^2 - 9a_4x^3 - 25a_5\cos 2x - 25a_6\sin 2x = 9x^3 + 5\cos 2x.$$

Then

$$\begin{cases} -9a_1 + 16a_3 = 0 \\ -9a_2 + 48a_4 = 0 \\ -9a_3 = 0 \\ -9a_4 = 9 \\ -25a_5 = 5 \\ -25a_6 = 0 \end{cases}.$$

The third equation ($-9a_3 = 0$) of the system gives $a_3 = 0$ and substituting for a_3 into the first equation ($-9a_1 + 16a_3 = 0$) one gets $-9a_1 = 0$, hence $a_1 = 0$.

The fourth equation ($-9a_4 = 9$) of the system gives $a_4 = -1$ and substituting for a_4 into the second equation ($-9a_2 + 48a_4 = 0$) one gets $-9a_2 - 48 = 0$, hence $a_2 = -\frac{48}{9} = -\frac{16}{3}$.

It follows from the fifth equation ($-25a_5 = 5$) of the system that $a_5 = -\frac{5}{25} = -\frac{1}{5}$.

It follows from the sixth equation ($-25a_6 = 0$) of the system that $a_6 = 0$.

Solution of the system is

$$\begin{cases} a_1 = 0 \\ a_2 = -\frac{16}{3} \\ a_3 = 0 \\ a_4 = -1 \\ a_5 = -\frac{1}{5} \\ a_6 = 0 \end{cases}$$

Then

$y_p = -\frac{16}{3}x - x^3 - \frac{1}{5}\cos 2x$ and

$$y = y_c + y_p = C_1e^x + C_2e^{-x} + C_3\cos 3x + C_4\sin 3x - x^3 - \frac{16}{3}x - \frac{1}{5}\cos 2x.$$

Answer:

$$y = y_c + y_p = C_1e^x + C_2e^{-x} + C_3\cos 3x + C_4\sin 3x - x^3 - \frac{16}{3}x - \frac{1}{5}\cos 2x.$$