Answer on Question #64773 – Math – Differential Equations

Question

Solve (D3−7D+12)y=x3−5x2

Solution

To solve a nonhomogeneous linear differential equation (DE)
$(D^3 - 7D + 12)y = x^3 - 5x^2$,
that is,
$$y''' - 7y' + 12y = x^3 - 5x^2 \qquad (1)$$
we must
1) find the complementary function y_c that is the general solution of the associated homogeneous DE
$$y''' - 7y' + 12y = 0;$$
2) find any particular solution y_p of the nonhomogeneous equation
$$y''' - 7y' + 12y = x^3 - 5x^2.$$
The general solution of the equation (1) is $y = y_c + y_p$.

1) To solve the associated homogeneous DE
$$y''' - 7y' + 12y = 0 \qquad (2)$$
we substitute into equation the solution as exponential function $y = e^{mx}$. We get
$$m^3e^{mx} - 7me^{mx} + 5e^{mx} = 0$$
or
$(m^3 - 7m + 12)e^{mx} = 0$.
Since e^{mx} is never zero for real values of *x*, the last equation is satisfied only when *m* is a solution or root of the third-degree polynomial equation
$$m^3 - 7m + 12 = 0.$$
This equation is called the auxiliary equation of the differential equation (2).
Analytical solution of this equation is a difficult task. Approximate numerical solution is
$m_1 \approx 1.63 - 1.00i$, $m_2 \approx 1.63 + 1.00i$, $m_3 \approx -3.27$.
Thus we have three different roots of the auxiliary equation and the general solution of the associated homogeneous DE is
$y_c = C_1e^{(1.63-1.00i)x} + C_2e^{(1.63+1.00i)x} + C_3e^{-3.27x}$.
Now we use Euler's formula $e^{i\beta} = \cos\beta + i\sin\beta$. Then we get
$$y_c = c_1e^{1.63x}\cos x + c_2e^{1.63x}\sin x + c_3e^{-3.27x}$$
where $c_1,\ c_2,\ c_3$ are constants.

2) The particular solution of the nonhomogeneous equation will be found in the following form:
$$y_p = Ax^3 + Bx^2 + Cx + D,$$
where $A,\ B,\ C,\ D$ are the undetermined coefficients.
Substituting this solution into the equation (1) and equating the coefficients of corresponding terms, we find $A,\ B,\ C,\ D$. Find derivatives y'_p, y''_p, y'''_p
$y'_p = 3Ax^2 + 2Bx + C$
$y''_p = 6Ax + 2B$
$y'''_p = 6A$
Substituting the derivatives into the equation (1)

$$6A-7(3Ax^2+2Bx+C)+12(Ax^3+Bx^2+Cx+D)=x^3-5x^2$$

Equating the coefficients of corresponding terms, we find

$$\begin{cases}12A=1,\\ -21A+12B=-5\\ -14B+12C=0\\ 6A-7C+12D=0\end{cases}$$

$$\begin{cases}A=\frac{1}{12}\approx 0.083,\\ -21\cdot 0.083+12B=-5\\ -14B+12C=0\\ 6\cdot 0.083-7C+12D=0\end{cases}$$

$$\begin{cases}12B=-5+1.743=-3{,}257\\ -14B+12C=0\\ 0.5-7C+12D=0\end{cases}$$

$$\begin{cases}B=-\frac{39}{144}\approx -0.271\\ 14\cdot 0.271+12C=0\\ \frac{1}{2}-7C+12D=0\end{cases}$$

$$\begin{cases}C\approx -0.316\\ \frac{1}{2}+7\cdot 0.316+12D=0\end{cases}$$

$$D\approx -0.226$$

So we have $A\approx 0.083$, $B\approx -0.271$, $C\approx -0.316$, $D\approx -0.226$ and finally we get the general solution of differential equation (1):

$$y=c_1e^{1.63x}\cos x+c_2e^{1.63x}\sin x+c_3e^{-3.27x}+0.083x^3-0.271x^2-0.316x-0.226.$$

Answer:

$$y=c_1e^{1.63x}\cos x+c_2e^{1.63x}\sin x+c_3e^{-3.27x}+0.083x^3-0.271x^2-0.316x-0.226.$$