Answer on Question #64772 – Math – Differential Equations

Question

Solve

$y''' - 7y' - 6y = 1 + x.$

Solution

Equation

$$(D^3 - 7D - 6)y = 1 + x,$$

that is,

$$y''' - 7y' - 6y = 1 + x$$

is a linear non-homogeneous ordinary differential equation of the third order.
First, we solve the corresponding characteristic equation:

$$k^3 - 7k - 6 = 0,$$
$$(k+1)(k^2 - k - 6) = 0,$$
$$(k+1)(k+2)(k-3) = 0,$$
$$k = -1, k = -2, k = 3.$$

Then the solution of the homogeneous differential equation is the complementary function

$$y_c(x) = c_1 e^{-x} + c_2 e^{-2x} + c_3 e^{3x},$$

where $c_1, c_2,$ and c_3 are any real constants.
The additional solution to the complementary function is a particular integral, which we looking for in the form of

$$y_p(x) = ax + b.$$

To find the unknown constants a and b we substitute y_p into the original equation:

$$(ax+b)''' - 7(ax+b)' - 6(ax+b) = 1 + x,$$
$$-7a - 6b - 6ax = 1 + x.$$

Now we equate the coefficients of equal powers of x:

$$-7a - 6b = 1$$

and

$$-6a = 1,$$

whence

$$a = -\frac{1}{6}, \text{ and } b = \frac{1}{36}.$$

So we have

$$y_p(x) = -\frac{1}{6}x + \frac{1}{36}.$$

Finally, the general solution to the given differential equation can be written as

$$y(x) = y_c(x) + y_p(x) = c_1 e^{-x} + c_2 e^{-2x} + c_3 e^{3x} - \frac{1}{6}x + \frac{1}{36}.$$

Answer: $y(x) = c_1 e^{-x} + c_2 e^{-2x} + c_3 e^{3x} - \frac{1}{6}x + \frac{1}{36}.$