Answer on Question #64621 – Math – Differential Equations

Question

Solve the following differential equation:
(D3−D2+3D+5)y=exsin2x

Solution

To solve the inhomogeneous linear differential equation (DE)

$y''' - y'' + 3y' + 5y = e^x \sin 2x$ (1)

we must

1) find the complementary function y_c that is the general solution of the associated homogeneous DE

$y''' - y'' + 3y' + 5y = 0;$

2) find any particular solution y_p of the nonhomogeneous equation

$y''' - y'' + 3y' + 5y = e^x \sin 2x.$

The general solution of the equation (1) is $y = y_c + y_p$.

1) To solve the associated homogeneous DE

$y''' - y'' + 3y' + 5y = 0$ (2)

we substitute into equation the solution as the exponential function $y = e^{mx}$. We get

$$m^3 e^{mx} - m^2 e^{mx} + 3me^{mx} + 5e^{mx} = 0$$

or

$(m^3 - m^2 + 3m + 5)e^{mx} = 0.$

Since e^{mx} is not zero for real values of *x*, the last equation has only the roots which are the solutions of the third-degree polynomial equation:

$$m^3 - m^2 + 3m + 5 = 0.$$

This equation is called the auxiliary equation of the differential equation (2).

It is obvious that one of the roots is $m = -1$. This can be verified by substituting $m = -1$ in equation

$(-1)^3 - (-1)^2 + 3(-1) + 5 = -1 - 1 - 3 + 5 = 0.$

To find other roots divide $m^3 - m^2 + 3m + 5$ by $m + 1$

$$\begin{array}{r|l} & m^2 - 2m + 5 \\ \hline m+1 & m^3 - m^2 + 3m + 5 \\ & m^3 + m^2 \\ \hline & -2m^2 + 3m \\ & -2m^2 - 2m \\ \hline & 5m + 5 \\ & 5m + 5 \\ \hline & 0 \end{array}$$

$y_p = xe^x(A\cos(2x) + B\sin(2x)),$

$y'_p = Ae^x \cos 2x + Axe^x \cos 2x - 2Axe^x \sin 2x + Be^x \sin 2x + Bxe^x \sin 2x + 2Bxe^x \cos 2x,$

$y''_p = Ae^x \cos 2x - 2Ae^x \sin 2x + Ae^x \cos 2x + Axe^x \cos 2x - 2Axe^x \sin 2x - 2Ae^x \sin 2x$
$- 2Axe^x \sin 2x -$

$$-4Axe^x\cos 2x + Be^x\sin 2x + 2Be^x\cos 2x + Be^x\sin 2x + Bxe^x\sin 2x + 2Bxe^x\cos 2x + 2Be^x\cos 2x +$$
$$+2Bxe^x\cos 2x - 4Bxe^x\sin 2x,$$

or

$$y''_p = 2Ae^x\cos 2x - 4Ae^x\sin 2x - 3Axe^x\cos 2x - 4Axe^x\sin 2x +$$
$$+2Be^x\sin 2x + 4Be^x\cos 2x - 3Bxe^x\sin 2x + 4Bxe^x\cos 2x$$

$$y'''_p = 2Ae^x\cos 2x - 4Ae^x\sin 2x - 4Ae^x\sin 2x - 8Ae^x\cos 2x - 3Ae^x\cos 2x - 3Axe^x\cos 2x$$
$$+6Axe^x\sin 2x - 4Ae^x\sin 2x - 4Ae^x\sin 2x - 8Axe^x\cos 2x + 2Be^x\sin 2x + 4Be^x\cos 2x + 4Be^x\cos 2x -$$
$$-8Be^x\sin 2x - 3Be^x\sin 2x - 3Bxe^x\sin 2x - 6Bxe^x\cos 2x + 4Be^x\cos 2x + 4Bxe^x\cos 2x - 8Bxe^x\sin 2x$$

or

$$y'''_p = -9Ae^x\cos 2x - 12Ae^x\sin 2x - 11Axe^x\cos 2x + 2Axe^x\sin 2x -$$
$$-9Be^x\sin 2x + 12Be^x\cos 2x - 11Bxe^x\sin 2x - 2Bxe^x\cos 2x$$

substitute the derivatives into the equation (1)

$$y''' - y'' + 3y' + 5y = -9Ae^x\cos 2x - 12Ae^x\sin 2x - \cancel{11Axe^x\cos 2x} + \cancel{2Axe^x\sin 2x} -$$
$$-9Be^x\sin 2x + 12Be^x\cos 2x - \cancel{11Bxe^x\sin 2x} - \cancel{2Bxe^x\cos 2x} - (2Ae^x\cos 2x - 4Ae^x\sin 2x -$$
$$-\cancel{3Axe^x\cos 2x} - \cancel{4Axe^x\sin 2x} + 2Be^x\sin 2x + 4Be^x\cos 2x - \cancel{3Bxe^x\sin 2x} + \cancel{4Bxe^x\cos 2x}) +$$
$$+3(Ae^x\cos 2x + \cancel{Axe^x\cos 2x} - \cancel{2Axe^x\sin 2x} + Be^x\sin 2x + \cancel{Bxe^x\sin 2x} + \cancel{2Bxe^x\cos 2x}) +$$
$$+5(\cancel{Axe^x\cos 2x} + \cancel{Bxe^x\sin 2x}) = e^x\sin 2x$$

Crossed out vanished terms and we have

$$-9Ae^x\cos 2x - 12Ae^x\sin 2x - 9Be^x\sin 2x + 12Be^x\cos 2x - (2Ae^x\cos 2x - 4Ae^x\sin 2x -$$
$$+2Be^x\sin 2x + 4Be^x\cos 2x) + 3(Ae^x\cos 2x + Be^x\sin 2x) = e^x\sin 2x$$

Equating the coefficients of corresponding terms, we find

$$\begin{cases} -9A + 12B - 2A - 4B + 3A = -6A + 6B = 0 \\ -12A - 9B + 4A - 2B + 3B = -8A - 8B = 1 \end{cases}$$

Then we have $A = B = -\frac{1}{16}$and finally we get the general solution of differential equation (1)

$$y = c_1e^{-x} + c_2e^x\cos 2x + c_3e^x\sin 2x - \frac{1}{16}xe^x(\cos 2x + \sin 2x)$$

Answer: $y = c_1e^{-x} + c_2e^x\cos 2x + c_3e^x\sin 2x - \frac{1}{16}xe^x(\cos 2x + \sin 2x)$.