Answer on Question #64618 – Math – Calculus

Question

1. A page is to contain 24 square inches of print. The margins at top and bottom are 1.5 inches, at the side is 1 inches. Find the most economical dimension of the page.

Solution

Let *x* and *y* be the length and width of the page (measured in inches).
Then area of print is

$$Area = (x - 2 \cdot 1.5)(y - 2 \cdot 1) = 24 \;\; => y = 2 + \frac{24}{x-3}.$$

The area of the page is

$$S = xy = x\left(2 + \frac{24}{x-3}\right)$$

Minimizing the area of the poster

$S'(x) = \left(2x + \frac{24x}{x-3}\right)' = 2 + 24\frac{x-3-x}{(x-3)^2} = 2 - \frac{72}{(x-3)^2};$

$S'(x) = 0 => \; 2 - \frac{72}{(x-3)^2} = 0, x > 3;$

$(x-3)^2 = 36, x = 9.$

Since

$$S''(x) = \left(2 - \frac{72}{(x-3)^2}\right)' = \frac{144}{(x-4)^3},$$

$$S''(9) = \frac{144}{(9-3)^3} = \frac{2}{3} > 0.$$

$S(x)$ takes the minimum at $x = 9$ on $(3, \infty)$:

$$y = 2 + \frac{24}{9-3} = 6 \; (inches).$$

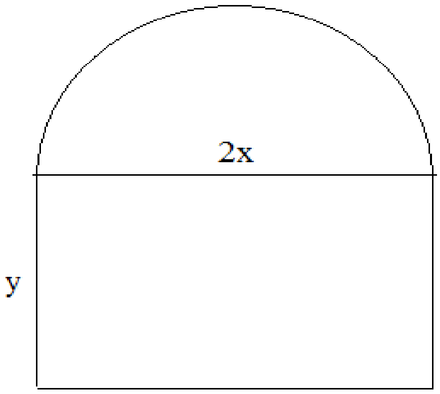
So the smallest page has dimension 9 inches × 6 inches.

Answer: 9 inches and 6 inches.

Question

2. A Norman window consists of a rectangle surrounded by a semi-circle. What shape gives the most light for a GIVEN perimeter?

Solution



Let x denote half the width of the rectangle (so x is the radius of the semicircle), and let y denote the height of the rectangle. Then the perimeter of the window is

$$P = 2y + 2x + \frac{1}{2}\pi 2x \Rightarrow y = \frac{P - (2+\pi)x}{2}.$$

The area of the window (which is proportional to the amount of light admitted) is

$$A = 2xy + \frac{1}{2}\pi x^2 = Px - \left(\frac{\pi}{2} + 2\right)x^2$$

To maximize the quantity we take the derivative and find the critical points:

$$A' = \left(Px - \left(\frac{\pi}{2} + 2\right)x^2\right)' = P - (\pi + 4)x;$$

$$A' = 0 \Rightarrow P - (\pi + 4)x = 0;$$

$$x = \frac{P}{\pi + 4}.$$

Since

$$A'' = (P - (\pi + 4)x)' = -(\pi + 4) < 0\,,$$

A takes the maximum at $x = \frac{P}{\pi+4}$.

Then

$$y = \frac{P - (2+\pi)\frac{P}{\pi + 4}}{2} = \frac{P}{\pi + 4}.$$

Hence, the window allowing maximal light in is the one with square base

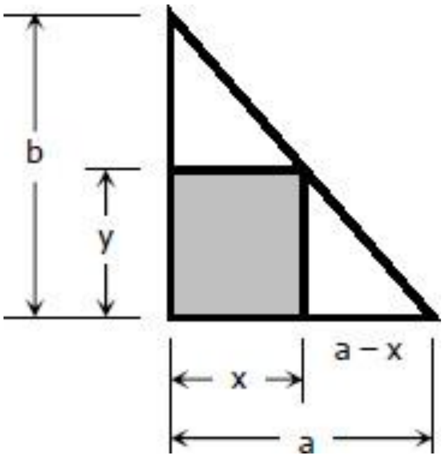
$$\frac{P}{\pi + 4} \times \frac{P}{\pi + 4}.$$

Answer: square.

Question

3. Find the dimension of the largest rectangular building that can be placed on a right triangular lot, facing one of the perpendicular sides.

Solution



Let x and y be the length and height of the building. Then area of lot is

$$\frac{1}{2}ab = xy + \frac{1}{2}x(b-y) + \frac{1}{2}y(a-x);$$

$$\frac{1}{2}ab = \frac{1}{2}ay + \frac{1}{2}bx;$$

$$ab = ay + bx \;\; => y = b - \frac{b}{a}x.$$

The area of the building is

$$S = xy = x\left(b - \frac{b}{a}x\right)$$

Minimizing the area of the poster

$S'(x) = \left(bx - \frac{b}{a}x^2\right)' = b - \frac{2b}{a}x;$

$S'(x) = 0 => \; b - \frac{2b}{a}x = 0, x > 0;$

$x = \frac{a}{2}.$

Since

$$S''(x) = \left(b - \frac{2b}{a}x\right)' = -\frac{2b}{a} < 0$$

$S(x)$ takes the maximum at $x = \frac{a}{2}$

$$y = b - \frac{ba}{2a} = \frac{b}{2}.$$

Hence, the dimension of the largest rectangular building

$$\frac{a}{2} \times \frac{b}{2}.$$

Answer: $\frac{a}{2}$ and $\frac{b}{2}$.