Answer on Question #64603 – Math – Calculus

Question

16. The top and bottom margins of a poster are each 6 cm, and the side margins are each 4 cm. If the area of the printed material on the poster (that is, the area between the margins) is fixed at 384 cm2, find the dimensions of the poster with the smallest total area.

Solution

Let *x* and *y* be the length and width of the poster (measured in cm). Then the printed area is

$$Area = (x - 2 \cdot 4)(y - 2 \cdot 6) = 384 \ \ => y = 12 + \frac{384}{x-8}.$$

The area of the poster is

$$S = xy = x\left(12 + \frac{384}{x-8}\right)$$

Minimizing the area of the poster

$$S'(x) = \left(12x + \frac{384x}{x-8}\right)' = 12 + 384\frac{x-8-x}{(x-8)^2} = 12 - \frac{3072}{(x-8)^2};$$

$$S'(x) = 0 => \ 12 - \frac{3072}{(x-8)^2} = 0, x > 8;$$

$$(x-8)^2 = 256, x = 16 + 8 = 24.$$

Since

$$\lim_{x \to 8^+} S(x) = \lim_{x \to \infty} S(x) = \infty,$$

$S(x)$ takes the minimum at *x* = 24 on (8, ∞).

So the smallest poster has dimension 24cm × 36cm.

Answer: 24cm × 36cm.

Question

17. Each rectangular page of a book must contain 30 cm2 of printed text, and each page must have 2 cm margins at top and bottom, and 1 cm margin at each side. What is the minimum possible area of such a page?

Solution

Let *x* and *y* be the length and width of the page (measured in cm). Then area of the printed text is

$$Area = (x - 2 \cdot 2)(y - 2 \cdot 1) = 30 \ \ => y = 2 + \frac{30}{x-4}.$$

The area of the page is

$$S = xy = x\left(2 + \frac{30}{x-4}\right)$$

Minimizing the area of the poster

$S'(x) = \left(2x + \frac{30x}{x-4}\right)' = 2 + 30\frac{x-4-x}{(x-4)^2} = 2 - \frac{120}{(x-4)^2};$

$S'(x) = 0 \Rightarrow 2 - \frac{120}{(x-4)^2} = 0, x > 4;$

$(x-4)^2 = 60, x = 2\sqrt{15} + 4.$

Since

$$S''(x) = \left(2 - \frac{120}{(x-4)^2}\right)' = \frac{240}{(x-4)^3}$$

$$S''\left(2\sqrt{15}+4\right) = \frac{240}{\left(2\sqrt{15}+4-4\right)^3} = \frac{2}{\sqrt{15}} > 0$$

$S(x)$ takes the minimum at $x = 2\sqrt{15} + 4$ on $(4, \infty)$.

$$y = 2 + \frac{30}{2\sqrt{15}+4-4} = 2 + \sqrt{15}\ (cm).$$

So the minimum possible area of such a page is

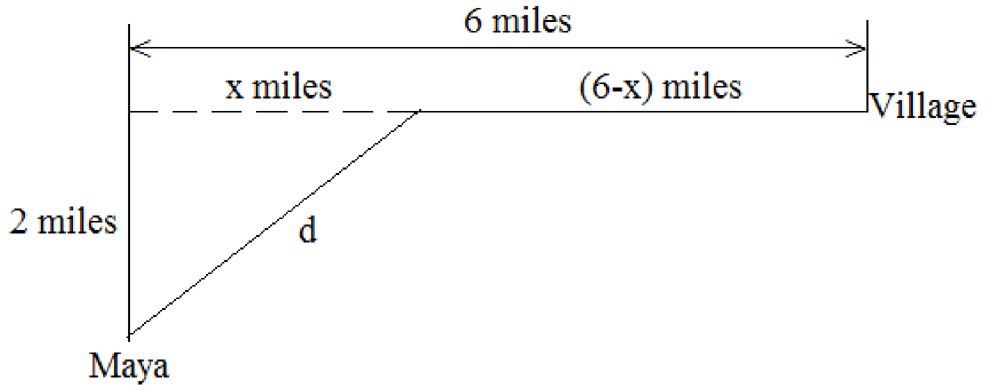
$$\left(2\sqrt{15}+4\right)\left(2+\sqrt{15}\right) = 36 + 8\sqrt{15}\ (cm^2).$$

Answer: $36 + 8\sqrt{15}\ cm^2$.

Question

18. Maya is 2 km offshore in a boat and wishes to reach a coastal village which is 6 km down a straight shoreline from the point on the shore nearest to the boat. She can row at 2 km/hr and run at 5 km/hr. Where should she land her boat to reach the village in the least amount of time?

Solution



Let *x* be the distance from the point on the shoreline nearest Maya's boat to the point where she lands her boat. Then she needs to row *d* miles at 2 mph and walk (6 – *x*) miles at 5 mph.

Using the Pythagorean Theorem

$$d^2 = x^2 + 2^2$$

The total amount of time to reach the village is

$$f(x) = \frac{\sqrt{x^2+4}}{2} + \frac{6-x}{5}, 0 \le x \le 6.$$

Minimizing the total amount of time

$$f'(x) = \left(\frac{\sqrt{x^2+4}}{2} + \frac{6-x}{5}\right)' = \frac{x}{2\sqrt{x^2+4}} - \frac{1}{5}.$$

$$f'(x) = 0 => \frac{x}{2\sqrt{x^2+4}} - \frac{1}{5} = 0;$$

$$5x = 2\sqrt{x^2+4};$$

$$25x^2 = 4x^2 + 16;$$

$$x^2 = \frac{16}{21};$$

$$x = \frac{4}{\sqrt{21}}, x \neq -\frac{4}{\sqrt{21}}.$$

Checking the endpoints and critical point, we have

$$f(0) = \frac{2}{2} + \frac{6}{5} = \frac{11}{5} = 2.2\ (hours), f(6) = \frac{\sqrt{6^2+4}}{2} + 0 = \sqrt{10}\ \approx 3.16(hours),$$

$$f\left(\frac{4}{\sqrt{21}}\right) = \frac{\sqrt{\frac{16}{21}+4}}{2} + \frac{6-\frac{4}{\sqrt{21}}}{5} = \frac{\sqrt{21}}{5} + \frac{6}{5} \approx 2.12\ (hours)$$

Minimum of $f(x)$ is attained at $x = \frac{4}{\sqrt{21}}$.

Answer: Maya should land her boat $\frac{4}{\sqrt{21}} \approx 0.87\ miles$ down the shoreline from the point nearest her boat.