Answer on Question #64601 – Math – Calculus

Question

Q1. The height of a rectangular box is increasing at a rate of 2 meters per second while the volume is decreasing at a rate of 5 cubic meters per second. If the base of the box is a square, at what rate is one of the sides of the base decreasing, at the moment when the base area is 64 square meters and the height is 8 meters?

Solution

The volume of the box with the square base is

$$V = h \cdot b^2.$$

The rate of change of volume equal to its time derivative

$$\frac{dV}{dt} = V'.$$

Find the rate of changing of the box volume:

$$V' = (h \cdot b^2)' = h' \cdot b^2 + 2hbb'$$

(here we used product rule

$$\big(f(x) \cdot g(x)\big) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$$

and chain rule

$$\Big(f\big(g(x)\big)\Big)' = f'\big(g(x)\big)g'(x)\,)$$

Find the rate of changing of the base side b'

$$V' = h' \cdot b^2 + 2hbb' \Rightarrow 2hbb' = V' - h' \cdot b^2 \Rightarrow b' = (V' - h' \cdot b^2)/2hb$$

where $h' = 2\ m/s$is the rate of the box height increasing, $V' = -5m^3/s$ is the rate of the box volume decreasing (minus sign indicates a decrease in volume)

$b = \sqrt{64} = 8$. Plug given values into

$$b' = (V' - h' \cdot b^2)/2hb$$

we get

$$b' = \frac{-5 - 2 \cdot 64}{2 \cdot 8 \cdot 8} = -\frac{5 + 128}{128} \approx -1.04m/s$$

(minus sign indicates a decrease in base side).

Answer: rate of the base side decreasing is $b' \approx -1.04m/s$.

Question

Q2. Sand is pouring out of a tube at 1 cubic meter per second. It forms a pile which has the shape of a cone. The height of the cone is equal to the radius of the circle at its base. How fast is the sandpile rising when it is 2 meters high?

Solution

The volume of the cone is equal to

$$V = \frac{1}{3}\pi R^2 h,$$

where R is the radius of the circle at its base, h is the height of the cone. Since by assumption of task the height of the cone is equal to the radius of the circle, i.e. $R = h$, then

$$V = \frac{1}{3}\pi h^3 = \frac{1}{3}\pi R^3.$$

The rate of change of volume equal to its time derivative

$$\frac{dV}{dt} = V'.$$

Find the rate of change of the volume of the sandpile:

$$V' = \left(\frac{1}{3}\pi h^3\right)' = \frac{1}{3}\pi \cdot 3h^2 h' = \pi h^2 h' = \pi h^2 R'.$$

Then the rate of increase in the height and radius of the cone is

$$h' = R' = V'/\pi h^2$$

Plug given values ($V' = 1\ m^3/\text{sec}$ and $h = 2\ m$) into last formula, we get the rate of increase in the height and radius of the cone

$$h' = R' = \frac{1}{4\pi} \approx 0{,}08\ \text{m/sec} = 8\ \text{cm/sec}.$$

Answer: the rate of increase of height and radius of the sandpile when it is 2 meters high, is $h' = R' = 0{,}08\ \text{m/sec} = 8\ \text{cm/sec}.$

Question

Q3. A water tank is in the shape of a cone with vertical axis and vertex downward. The tank has radius 3 m and is 5 m high. At first the tank is full of water, but at time t = 0 (in seconds), a small hole at the vertex is opened and the water begins to drain. When the height of water in the tank has dropped to 3 m, the water is flowing out at 2 m3/s. At what rate, in meters per second, is the water level dropping then?

Solution

Volume of a cone is

$$V = \frac{1}{3}\pi R^2 h$$

where R is the radius of the cone, h is the height. We know height of the cone and radius of the cone at the top so we can find the radius for the any height using a proportion of similar triangles. The ratio of the height of the tank to the radius:

$$\frac{R}{h} = \frac{3}{5}$$

When the height of the water is h meters the radius is:

$$R = \frac{3}{5}h$$

Plug this into the $V = \frac{1}{3}\pi R^2 h$, we get

$$V = \frac{1}{3}\pi\left(\frac{3}{5}h\right)^2 h = \frac{3\pi}{25}h^3$$

The rate of change of volume equal to its time derivative $\frac{dV}{dt} = V'$

Take derivative of each side

$$V' = \frac{9\pi}{25}h^2 h'$$

Plug numbers:

$$V' = \frac{9\pi}{25}3^2 h' = 2\mathrm{m}^3/\mathrm{sec}$$

Then

$$h' = \frac{2 \cdot 25}{81\pi} \approx 0.2\mathrm{m/sec}$$

Answer: the rate of decreasing water level is $h' \approx 0.2\mathrm{m/sec}$.