Answer on Question #64536–Math–Real Analysis

Question

Show that

if $z_n = (a^n + b^n)^{\frac{1}{n}}$

where $0 < a < b$, then

$$\lim_{n\to\infty} z_n = b .$$

Solution

Note that

$$z_n = (a^n + b^n)^{\frac{1}{n}} = \left(b^n \cdot (\frac{a^n}{b^n} + 1)\right)^{\frac{1}{n}} = (b^n)^{\frac{1}{n}} \cdot \left(\frac{a^n}{b^n} + 1\right)^{\frac{1}{n}} = b \cdot \left(\frac{a^n}{b^n} + 1\right)^{\frac{1}{n}} = b \cdot \left(\left(\frac{a}{b}\right)^n + 1\right)^{\frac{1}{n}} .$$

For $0 < a < b$ we get $0 < \frac{a}{b} < 1$ hence

$$\lim_{n\to\infty} \left(\frac{a}{b}\right)^n = 0 .$$

$$\lim_{n\to\infty} z_n = \lim_{n\to\infty} (a^n + b^n)^{\frac{1}{n}} = \lim_{n\to\infty} b \cdot \left(\left(\frac{a}{b}\right)^n + 1\right)^{\frac{1}{n}} = b \lim_{n\to\infty} \left(\left(\frac{a}{b}\right)^n + 1\right)^{\frac{1}{n}} = b \cdot (0+1)^0 = b \cdot 1 = b.$$

Answer: b .