Answer on Question #64534 – Math – Real Analysis

Question

If $a > 0, b > 0$, show that $\lim_{n\to\infty}\left(\sqrt{(n+a)(n+b)} - n\right) = \frac{a+b}{2}$.

Solution

Method 1

$$\lim_{n\to\infty}\left(\sqrt{(n+a)(n+b)} - n\right) = \lim_{n\to\infty}\frac{\sqrt{(n+a)(n+b)} - n}{\sqrt{(n+a)(n+b)} + n}$$

$$\cdot\left(\sqrt{(n+a)(n+b)} + n\right) = \lim_{n\to\infty}\frac{(n+a)(n+b) - n^2}{\sqrt{(n+a)(n+b)} + n} =$$

$$= \lim_{n\to\infty}\frac{n^2 + bn + an + ab - n^2}{\sqrt{(n+a)(n+b)} + n} = \lim_{n\to\infty}\frac{(a+b)n + ab}{\sqrt{(n+a)(n+b)} + n} =$$

$$= \lim_{n\to\infty}\frac{n\left((a+b) + \frac{ab}{n}\right)}{n\left(\sqrt{\left(1+\frac{a}{n}\right)\left(1+\frac{b}{n}\right)} + 1\right)} =$$

$$= \lim_{n\to\infty}\frac{(a+b) + \frac{ab}{n}}{\sqrt{\left(1+\frac{a}{n}\right)\left(1+\frac{b}{n}\right)} + 1} =$$

$$= \frac{\lim_{n\to\infty}\left((a+b) + \frac{ab}{n}\right)}{\lim_{n\to\infty}\left(\sqrt{\left(1+\frac{a}{n}\right)\left(1+\frac{b}{n}\right)} + 1\right)} = \frac{(a+b)+0}{\sqrt{(1+0)(1+0)}+1} = \frac{a+b}{1+1} = \frac{a+b}{2}.$$

QED

Method 2

Transforming the expression $\sqrt{(n+a)(n+b)} - n$ we obtain

$$\sqrt{(n+a)(n+b)}-n=\sqrt{n^2+n(a+b)+ab}-n=\sqrt{n^2\left(1+\frac{a+b}{n}+\frac{ab}{n^2}\right)}-n=$$

$$=n\sqrt{1+\frac{a+b}{n}+\frac{ab}{n^2}}-n=n\left(\sqrt{1+\frac{a+b}{n}+\frac{ab}{n^2}}-1\right).$$

Since $\left(1+\alpha(n)\right)^{\beta}-1\sim\beta\cdot\alpha(n)$ where $\lim_{n\to\infty}\alpha(n)=0$, we have:

$$\lim_{n\to\infty}\left(\sqrt{(n+a)(n+b)}-n\right)=\lim_{n\to\infty}n\left(\sqrt{1+\frac{a+b}{n}+\frac{ab}{n^2}}-1\right)=\lim_{n\to\infty}n\cdot\frac{1}{2}\cdot\left(\frac{a+b}{n}+\frac{ab}{n^2}\right)$$

$$=\lim_{n\to\infty}\left(\frac{a+b}{2}+\frac{ab}{2n}\right)=\frac{a+b}{2}+0=\frac{a+b}{2}.$$

QED.