Answer on Question #64529 – Math – Real Analysis

Question

Let $y_n = \sqrt{n+1} - \sqrt{n}$ for $n \in \mathbb{N}$. Show that y_n and $\sqrt{n} \cdot y_n$ converge. Find their limits.

Solution

$$y_n = \sqrt{n+1} - \sqrt{n} = \frac{(\sqrt{n+1}-\sqrt{n})(\sqrt{n+1}+\sqrt{n})}{\sqrt{n+1}+\sqrt{n}} = \frac{n+1-n}{\sqrt{n+1}+\sqrt{n}} = \frac{1}{\sqrt{n+1}+\sqrt{n}} < \frac{1}{\sqrt{n}},\ n \in \mathbb{N}.$$

Obviously $\frac{1}{\sqrt{n}} \to 0$ as $n \to \infty$ because for all $\varepsilon > 0\ \exists\ N = N(\varepsilon)$: $\forall n > N(\varepsilon)\ \frac{1}{\sqrt{n}} < \varepsilon$ (for example $N(\varepsilon) = \left[\frac{1}{\varepsilon^2}\right] + 1$ where $\left[\frac{1}{\varepsilon^2}\right]$ denotes the integer part of $\frac{1}{\varepsilon^2}$).

So we have

$$0 < \frac{1}{\sqrt{n+1}+\sqrt{n}} < \frac{1}{\sqrt{n}}.$$

Using squeeze theorem we obtain that

$$\frac{1}{\sqrt{n+1}+\sqrt{n}} \to 0 \text{ as } n \to \infty.$$

Thus $\lim_{n\to\infty} y_n = \lim_{n\to\infty}\left(\sqrt{n+1} - \sqrt{n}\right) = 0.$

Next,

$$\sqrt{n} \cdot y_n = \frac{\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} = \frac{\sqrt{n}}{\sqrt{n\left(1+\frac{1}{n}\right)}+\sqrt{n}} = \frac{\sqrt{n}}{\sqrt{n}\left(\sqrt{1+\frac{1}{n}}+1\right)} = \frac{1}{\sqrt{1+\frac{1}{n}}+1}.$$

Obviously $\frac{1}{n} \to 0$ as $n \to \infty$ because for all $\varepsilon > 0\ \exists\ N = N(\varepsilon)$: $\forall n > N(\varepsilon)\ \frac{1}{n} < \varepsilon$ (for example, $N(\varepsilon) = \left[\frac{1}{\varepsilon}\right] + 1$, where $\left[\frac{1}{\varepsilon}\right]$ denotes the integer part of $\frac{1}{\varepsilon}$).

Then

$$\frac{1}{\sqrt{1+\frac{1}{n}}+1} \to \frac{1}{\sqrt{1}+1} = \frac{1}{2} \text{ as } n \to \infty.$$

Thus $\lim_{n\to\infty} \sqrt{n} \cdot y_n = \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} = \frac{1}{2}.$

Answer: $\lim_{n\to\infty} y_n = 0;\ \lim_{n\to\infty} \sqrt{n} \cdot y_n = \frac{1}{2}.$