Answer on Question #64525 – Math – Real Analysis

Question

Show that the following sequences are not convergent:
(a) 2^n, **(b)** $(-1)^n n^2$.

Solution

(a) Assume the contrary:

$$lim 2^n = a \text{ exists. } \textbf{(1)}$$

Then $\forall \varepsilon > 0\ \exists K \in N$ such that

$$|2^n - a| < \varepsilon \ \ \forall n > K.$$

If we put $\varepsilon = 1$ we have:

$$|2^n - a| < 1 \ \ \forall n > K.$$

Hence $2^n < a + 1$. It is known that $n < 2^n$, so

$$n < 2^n < a + 1, \textbf{(2)}$$
$$n > K.$$

Since $a + 1 \in R$, by the Archimedean Property, $\exists M \in N$ such that $M > a + 1$.
Then $\forall n > max(K, M)$ one gets

$$n > a + 1. \ \textbf{(3)}$$

There is a contradiction between (2) and (3). It means that assumption (1) is false.
Hence 2^n is not convergent.

(b) Assume the contrary:

$$lim(-1)^n n^2 = a \text{ exists.} \qquad \textbf{(4)}$$

Then $\forall \varepsilon > 0\ \exists K \in N$ such that $|(-1)^n n^2 - a| < \varepsilon \ \ \forall n > K$.
If we put $\varepsilon = \frac{1}{2}$ we have: $|(-1)^n n^2 - a| < \frac{1}{2} \ \ \forall n > K$.
In particular, $|(-1)^{2n} 4n^2 - a| < \frac{1}{2}$ and $|(-1)^{2n+1}(2n+1)^2 - a| < \frac{1}{2}$.
So $|4n^2 - a| < \frac{1}{2}$ and $|4n^2 + 4n + 1 + a| < \frac{1}{2}$, $\forall n > K$.
Hence
$|4n^2 - a| + |4n^2 + 4n + 1 + a| < \frac{1}{2} + \frac{1}{2} = 1$,
That is,

$$|4n^2 - a| + |4n^2 + 4n + 1 + a| < 1 \qquad \textbf{(5)}$$

Using the triangle inequality,

$$|4n^2 - a| + |4n^2 + 4n + 1 + a| \geq |(4n^2 - a) + (4n^2 + 4n + 1 + a)| =$$
$$= |8n^2 + 4n + 1| > 1,$$

because $n > K \geq 1$.
That is,

$$|4n^2 - a| + |4n^2 + 4n + 1 + a| > 1 \qquad \textbf{(6)}$$

There is a contradiction between (5) and (6).
It means that assumption (4) is false.
Hence $(-1)^n n^2$ is not convergent.