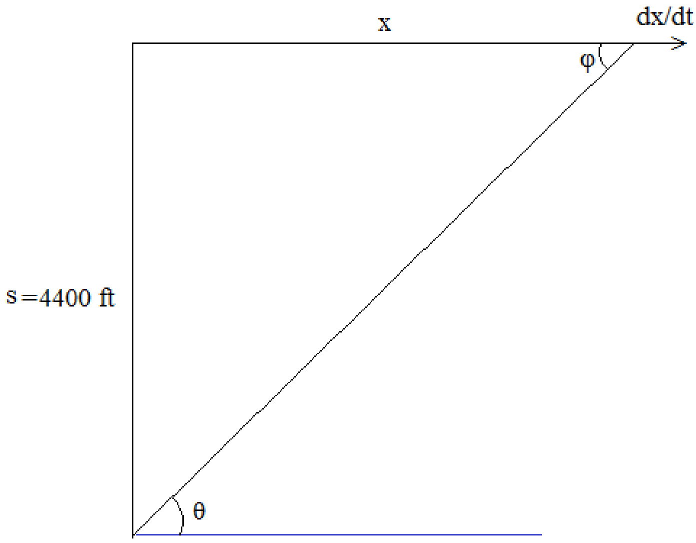
Answer on Question #64138 – Math – Geometry

Question

1. An airplane at an altitude of 4400ft is flying horizontally away from an observer. At the instant when the angle of elevation is 45 degrees, the angle is decreasing at the rate of .05 rad/sec. How fast is the airplane flying at the instant?

Solution



We have that

$$\cot\varphi = \frac{x}{s}, \varphi = \theta => x = s\cot\theta.$$

Differentiate both sides using Product rule and Chain rule

$$\frac{dx}{dt} = (s\cot\theta)' = \frac{ds}{dt}\cot\theta - s\cdot\frac{1}{\sin^2\theta}\cdot\frac{d\theta}{dt}.$$

Since $s = 4400\ ft = const, \frac{ds}{dt} == 0$. Therefore

$$\frac{dx}{dt} = -s\cdot\frac{1}{\sin^2\theta}\cdot\frac{d\theta}{dt}.$$

At the instant

$$s = 4400 ft, \theta = 45°, \frac{d\theta}{dt} = -0.05\ rad/sec.$$

The negative sign indicates that the angle θ is decreasing.

Then

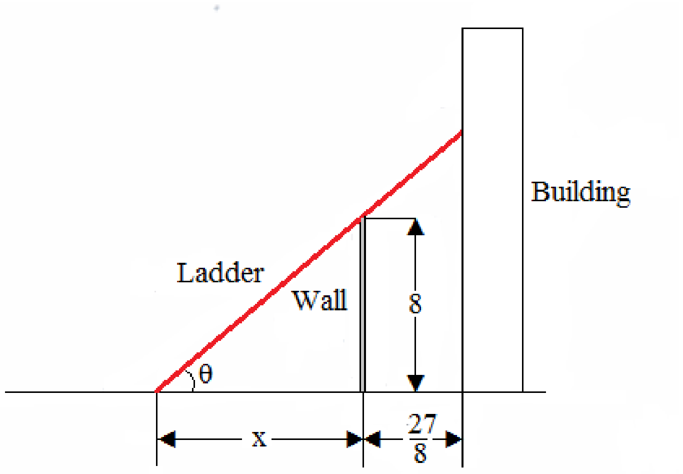
$$\frac{dx}{dt} = -4400\cdot\frac{1}{\sin^2 45°}\cdot(-0.05) = 440\ (ft/sec).$$

Answer: $440\ (ft/sec)$.

Question

2. A wall 8ft high is 27/8ft from a building. Find the length of the shortest ladder which will clear the wall and rest with one end on the ground and the other end on the building. Also, find the angle which this ladder makes with the horizontal.

Solution



We have that

$$L = \frac{x + \frac{27}{8}}{\cos\theta},$$

where

$$\tan\theta = \frac{8}{x}.$$

Since

$$1 + \tan^2\theta = \frac{1}{\cos^2\theta}, 0 < \theta < \frac{\pi}{2},$$

then

$$\cos\theta = \frac{1}{\sqrt{1 + \tan^2\theta}} = \frac{1}{\sqrt{1 + \left(\frac{8}{x}\right)^2}} = \frac{x}{\sqrt{64 + x^2}}.$$

Therefore

$$L(x) = \left(x + \frac{27}{8}\right)\frac{\sqrt{64 + x^2}}{x}.$$

Find the first derivative using Product rule and Chain rule

$$L'(x) = \left(\left(x + \frac{27}{8}\right)\frac{\sqrt{64 + x^2}}{x}\right)' = \frac{\sqrt{64 + x^2}}{x} + \left(x + \frac{27}{8}\right)\frac{\frac{2x \cdot x}{2\sqrt{64 + x^2}} - \sqrt{64 + x^2}}{x^2} =$$

$$= \frac{\sqrt{64 + x^2}}{x} + \left(x + \frac{27}{8}\right)\frac{x^2 - 64 - x^2}{x^2\sqrt{64 + x^2}} = \frac{x(64 + x^2) - 64x - 27 \cdot 8}{x^2\sqrt{64 + x^2}} =$$

$$= \frac{x^3 - 216}{x^2\sqrt{64 + x^2}}.$$

$$L'(x) = 0 => \frac{x^3 - 216}{x^2\sqrt{64 + x^2}} = 0 => x^3 - 216 = 0, x > 0.$$
$(x - 6)(x^2 + 6x + 36) = 0;$
The only root is $x = 6$, because $x^2 + 6x + 36 > 0$ for all real numbers x.
If $0 < x < 6$, then $L'(x) > 0$. If $x > 6$, then $L'(x) < 0$. Therefore minimum occurs at $x = 6$.

$$\tan\theta = \frac{8}{6} = \frac{4}{3}, \theta = \tan^{-1}\frac{4}{3}.$$

$$\cos\theta = \frac{1}{\sqrt{1 + \tan^2\theta}} = \frac{1}{\sqrt{1 + \left(\frac{4}{3}\right)^2}} = \frac{3}{5};$$

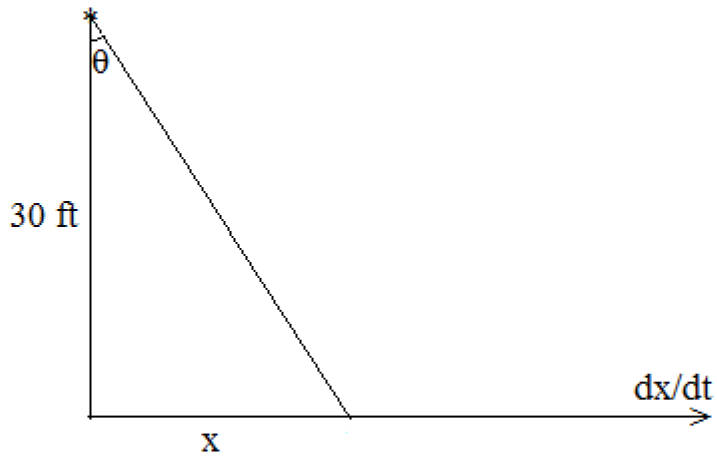
$$L = \frac{6 + \frac{27}{8}}{\frac{3}{5}} = \frac{125}{8}.$$

Answer: $\frac{125}{8}$; $\tan^{-1}\frac{4}{3}$.

Question

3. A man is walking along a sidewalk at the rate of 5ft/sec. A searchlight on the ground 30ft from the walk is kept trained on him. At what rate is the searchlight revolving when the man is 20ft away from the point on the sidewalk nearest the light?

Solution



We have that

$$\tan\theta = \frac{x}{30} => \theta = \tan^{-1}\frac{x}{30}.$$

Differentiate both sides using Chain rule

$$\frac{d\theta}{dt} = \frac{1}{1 + \left(\frac{x}{30}\right)^2} \cdot \left(\frac{x}{30}\right)';$$

$$\frac{d\theta}{dt} = \frac{1}{1 + \left(\frac{x}{30}\right)^2} \cdot \frac{1}{30} \cdot \frac{dx}{dt}.$$

When $x = 20\ ft$ and $\frac{dx}{dt} = 5\ ft/sec$

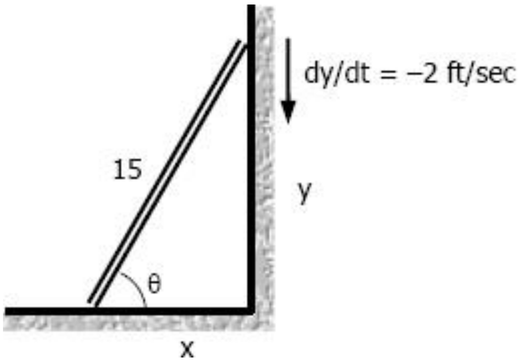
$$\frac{d\theta}{dt} = \frac{1}{1 + \left(\frac{20}{30}\right)^2} \cdot \frac{1}{30} \cdot 5 = \frac{3}{26}\ (rad/sec).$$

Answer: $\frac{3}{26}\ (rad/sec)$.

Question

4. A ladder 15ft long leans against a vertical wall. If the top slides down at 2ft/sec, how fast is the angle of elevation of the ladder decreasing, when the lower end is 12ft from the wall?

Solution



$$\sin\theta = \frac{y}{15} \Rightarrow\ \theta = \sin^{-1}\frac{y}{15}$$

Differentiate both sides

$$\frac{d\theta}{dt} = \left(\sin^{-1}\frac{y}{15}\right)'.$$

Use Chain rule

$$\frac{d\theta}{dt} = \frac{1}{\sqrt{1 - \left(\frac{y}{15}\right)^2}} \cdot \left(\frac{y}{15}\right)';$$

$$\frac{d\theta}{dt} = \frac{1}{\sqrt{225 - y^2}} \cdot \frac{dy}{dt}.$$

When $x = 12\ ft$, compute

$$y = \sqrt{15^2 - x^2} = \sqrt{225 - 12^2} = 9\ (ft).$$

Then

$$\frac{d\theta}{dt} = \frac{1}{\sqrt{225 - 9^2}} \cdot (-2) = -\frac{1}{6}\ (rad/sec).$$

The negative sign indicates that the angle is decreasing.

Answer: $-\frac{1}{6}\ (rad/sec)$.