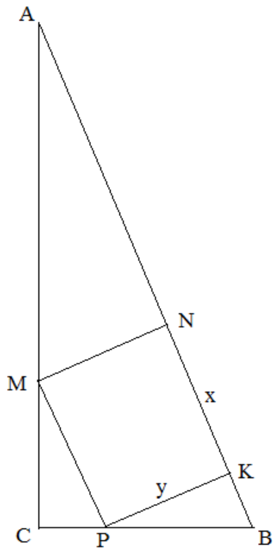
Answer on Question #64136 – Math – Geometry

Question

2. A right triangle has hypotenuse of length 13 and one leg of length 5. Find the dimensions of the rectangle of largest areas which has one side along the hypotenuse and the ends of the opposite side on the legs of this triangle.

Solution



$AB = 13, AC = 12, BC = \sqrt{AB^2 - AC^2} = \sqrt{13^2 - 12^2} = 5,$
$MNKP$ is rectangle, $MN = KP = y, NK = MP = x.$

The rectangle creates three small triangles inside the big triangle. These are all similar to the big triangle by Angle-Angle similarity.

$\frac{CP}{MP} = \frac{BC}{AB}, \frac{BP}{KP} = \frac{AB}{AC}$, hence

$$\frac{CP}{x} = \frac{5}{13}, \frac{BP}{y} = \frac{13}{12}.$$

Therefore

$$BC = 5 = CP + BP = \frac{5x}{13} + \frac{13y}{12} => y = \frac{12}{13}\left(5 - \frac{5x}{13}\right) =>$$

$$y = \frac{780 - 60x}{169}.$$

Then the area of the rectangle is

$$A_r = xy = \frac{780x - 60x^2}{169}.$$

Find the first derivative

$$A_r' = \left(\frac{780x - 60x^2}{169}\right)' = \frac{780 - 120x}{169}.$$

$$A_r' = 0 => \frac{780 - 120x}{169} = 0 => x = \frac{780}{120} => x = 6.5.$$

Find the second derivative

$$A_r'' = \left(\frac{780 - 120x}{169}\right)' = \frac{-120}{169} < 0 => \text{maximum is attained at } x = 6.5.$$

$$y = \frac{780 - 60 \cdot 6.5}{169} = \frac{30}{13}.$$

Answer: $6.5 \times \frac{30}{13}$.

Question

3. A closed box with a square base is to have a volume of 2000 cu. Inches. The material for the top and bottom of the box is to cost $3 per square inch, and the material for the sides is the cost $1.50 per square inch. If the cost of the material is to be the least, find the dimensions of the box.

Solution

Let Length=x and Height=y. Then

$$V = x^2 y = 2000\ (in^3)$$

The cost of the material depends on the square of each side

$$C = 2x^2 \cdot 3 + 4xy \cdot 1.5$$

We have that

$$y = \frac{V}{x^2}.$$

Then

$$C(x) = 6x^2 + 6x\frac{V}{x^2} = 6x^2 + 6\frac{V}{x}.$$

Minimize the cost

$$C'(x) = \left(6x^2 + 6\frac{V}{x}\right)' = 12x - 6\frac{V}{x^2}$$

$$C'(x) = 0 => 12x - 6\frac{V}{x^2} = 0;$$

$$x^3 - \frac{V}{2} = 0, x > 0;$$

$$\left(x - \sqrt[3]{\frac{V}{2}}\right)\left(x^2 + x\sqrt[3]{\frac{V}{2}} + \sqrt[3]{\frac{V^2}{4}}\right) = 0;$$

$$x = \sqrt[3]{\frac{V}{2}}, x^2 + x\sqrt[3]{\frac{V}{2}} + \sqrt[3]{\frac{V^2}{4}} > 0\ for\ x \in R.$$

Find the second derivative

$$C''(x) = \left(12x - 6\frac{V}{x^2}\right)' = 12 + 12\frac{V}{x^3};$$

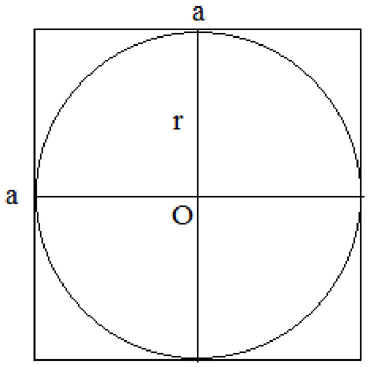
$$C''\left(\sqrt[3]{\frac{V}{2}}\right) = 12 + 12\frac{V}{\frac{V}{2}} = 36 > 0 => \text{minimum is attained at } x = \sqrt[3]{\frac{V}{2}}.$$

$$x = \sqrt[3]{\frac{2000}{2}} = 10\ (in.), y = \frac{V}{x^2} = \frac{2000}{10^2} = 20\ (in.).$$

Answer: 10 in., 20 in.

Question

1. Find the dimensions of the largest circle that can be inscribed in a square of 12 inches.



Solution

Let the side of the square be equal 12 inches: a=12 in. Then the radius of the largest circle that can be inscribed in a square

$$r = \frac{a}{2} = \frac{12}{2} = 6(in.)$$

Answer: 6 in.

Question

1. A manufacturer makes aluminum cups of a given volume (16 in3) in the form of right circular cylinders open at the top. Find the dimensions which use the least material.

Solution

Let radius = R and altitude = H. Then

$$V = \pi R^2 H = 16\ (in^3) => H = \frac{V}{\pi R^2}$$

The cost of the material depends on the total area

$$A = \pi R^2 + 2\pi RH = \pi R^2 + 2\pi R \frac{V}{\pi R^2} = \pi R^2 + 2\frac{V}{R} = A(R).$$

Minimize the total area

$$A'(R) = \left(\pi R^2 + 2\frac{V}{R}\right)' = 2\pi R - 2\frac{V}{R^2}.$$

$$A'(R) = 0 \ => 2\pi R - 2\frac{V}{R^2} = 0;$$

$$R^3 = \frac{V}{\pi};$$

$$\left(R - \sqrt[3]{\frac{V}{\pi}}\right)\left(R^2 + R\sqrt[3]{\frac{V}{\pi}} + \sqrt[3]{\frac{V^2}{\pi^2}}\right) = 0;$$

$R = \sqrt[3]{\frac{V}{\pi}}, \quad R^2 + R\sqrt[3]{\frac{V}{\pi}} + \sqrt[3]{\frac{V^2}{\pi^2}} > 0, R > 0.$

Find the second derivative:

$$A''(R) = \left(2\pi R - 2\frac{V}{R^2}\right)' = 2\pi + 4\frac{V}{R^3}$$

$$A''\left(\sqrt[3]{\frac{V}{\pi}}\right) = 2\pi + 4\frac{V}{\left(\sqrt[3]{\frac{V}{\pi}}\right)^3} = 2\pi + 4\pi > 0 =>$$

minimum is attained at $R = \sqrt[3]{\frac{V}{\pi}}$.

$$R = \sqrt[3]{\frac{V}{\pi}} = \sqrt[3]{\frac{16}{\pi}} = 2\sqrt[3]{\frac{2}{\pi}} \approx 1.72\ (in.),$$

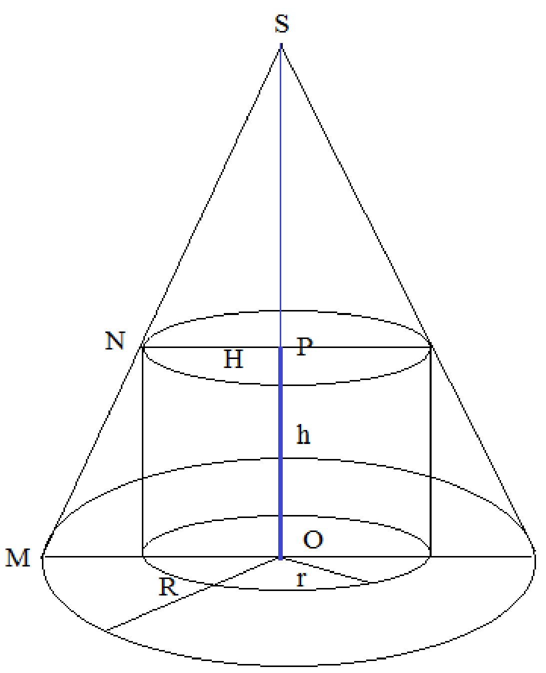
$$H = \frac{V}{\pi R^2} = \frac{VR}{\pi R^3} = \frac{V\sqrt[3]{\frac{V}{\pi}}}{V} = \sqrt[3]{\frac{V}{\pi}} = \sqrt[3]{\frac{16}{\pi}} = 2\sqrt[3]{\frac{2}{\pi}} \approx 1.72(in.).$$

Answer: $R = H = \sqrt[3]{\frac{16}{\pi}}$ in.

Question

2. Find the dimensions of the right circular cylinder of maximum volume which can be inscribed in a right circular cone of altitude 10 and radius 12.

Solution



We have that

$H = 10, R = 12.$

ΔSOM is similar to ΔSPN by Angle-Angle similarity. Then

$$\frac{OS}{SP} = \frac{OM}{NP} \Rightarrow \frac{H}{H-h} = \frac{R}{r} \Rightarrow H - h = \frac{rH}{R} \Rightarrow h = H - \frac{rH}{R} \Rightarrow$$

$$\Rightarrow h = H\frac{R-r}{R} = H\left(1 - \frac{r}{R}\right).$$

The volume of the cylinder is

$$V_{cyl} = \pi r^2 h = \pi r^2 H\left(1 - \frac{r}{R}\right) = \pi H\left(r^2 - \frac{r^3}{R}\right) = V(r).$$

Maximize the volume

$$V'(r) = \left(\pi H\left(r^2 - \frac{r^3}{R}\right)\right)' = \pi H\left(2r - \frac{3r^2}{R}\right).$$

$$V'(r) = 0 \Rightarrow \pi H\left(2r - \frac{3r^2}{R}\right) = 0;$$

$r = 0$ or $r = \frac{2}{3}R.$

Find the second derivative

$$V''(r) = \left(\pi H\left(2r - \frac{3r^2}{R}\right)\right)' = \pi H\left(2 - \frac{6r}{R}\right);$$

$V''\left(\frac{2R}{3}\right) = \pi H\left(2 - \frac{6\cdot\frac{2R}{3}}{R}\right) = -4\pi H < 0 \Rightarrow$ maximum is attained at $r = \frac{2}{3}R$, hence $\frac{r}{R} = \frac{2}{3}.$

$V''(0) = \pi H > 0.$

$$r = \frac{2}{3}R = \frac{2}{3}\cdot 12 = 8, \qquad h = H\left(1 - \frac{r}{R}\right) = H\left(1 - \frac{2}{3}\right) = \frac{1}{3}H = \frac{1}{3}\cdot 10 = \frac{10}{3}.$$

Answer: $r = 8$, $h = \frac{10}{3}$.