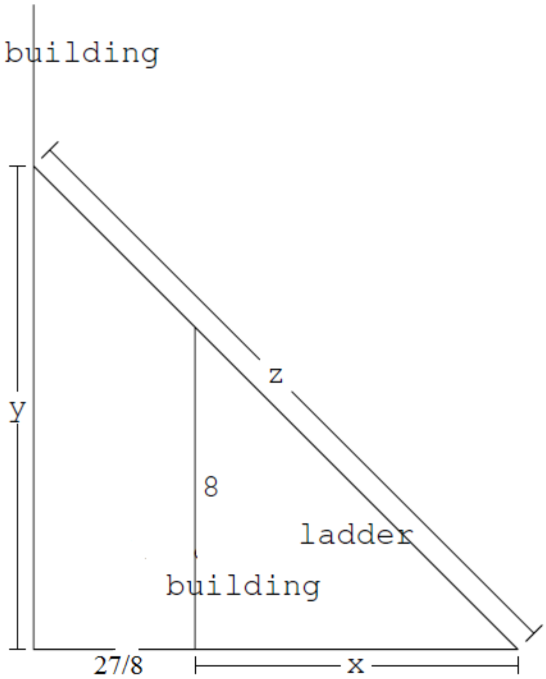
Answer on Question #63950 – Math – Analytic Geometry

Question

A building 8 feet high is 27/8 feet from a building. Find the length of the shortest ladder which will clear the wall and rest with one end on the ground and the other end on the building. Also, find the angle which this ladder makes with the horizontal?

Solution



From the picture we see that we have a right triangle involving the hypotenuse labeled by z, which represents the length of the ladder. So from the Pythagorean Theorem we have the equation

$$z^2 = y^2 + \left(\frac{27}{8} + x\right)^2$$

We also have similar right triangles, one with vertical side 8 feet, the height of the fence, and the other with vertical side y feet, where y indicates how far up the building the ladder touches. Using the fact that ratios of sides for similar triangles are equal we have

$$\frac{y}{8} = \frac{\frac{27}{8} + x}{x}$$

We'll solve this equation for y and substitute it in the equation above involving the length of the ladder, z.

$$y = \frac{27 + 8x}{x}$$

$$z^2 = \left(\frac{27 + 8x}{x}\right)^2 + \left(\frac{27}{8} + x\right)^2$$

At this point, most of students solved for z (since that's what we want to minimize) and then took the derivative. And the derivative is a mess. We can avoid most of the mess by observing that if z is minimized, then the quantity z^2 will also be minimized. So consider the function

$$L(x) = \left(\frac{27+8x}{x}\right)^2 + \left(\frac{27}{8} + x\right)^2$$

(which is z^2, the square of the length of the ladder), and we'll minimize this function.

$$L'(x) = 2\left(\frac{27+8x}{x}\right)\left(-\frac{27}{x^2}\right) + 2\left(\frac{27}{8} + x\right).$$

There is a critical point at x = 0 (L'(0) DNE). This is an endpoint of our domain, since we want the length of the piece labeled by x to have a positive length, and x = 0 corresponds to placing the ladder vertically against the fence. In this case, the ladder will never reach the building (our building is tall, but not that tall). So we solve for $L'(x)$ =0.

$$2\left(\frac{27+8x}{x}\right)\left(-\frac{27}{x^2}\right) + 2\left(\frac{27}{8} + x\right) = 0$$

$$\left(\frac{27}{8} + x\right) = \frac{1}{8}(27+8x) = \left(\frac{27+8x}{x}\right)\left(\frac{27}{x^2}\right)$$

As long as $x \neq -\frac{27}{8}$, we can divide both sides of this equation by the factor $(27+8x)$. And since the domain that we are interested does not contain negative numbers, we assume that $x \neq -\frac{27}{8}$. So we continue and get

$$\frac{1}{8} = \left(\frac{1}{x}\right)\left(\frac{27}{x^2}\right)$$

$$x^3 = 27 \cdot 8$$

$$x = \sqrt[3]{27 \cdot 8} = 6\ ft.$$

We have one critical point in the domain at x=6. We perform the first derivative test to check that it is a minimum: $L'(4) \approx -35$ and $L'(8) \approx 13$. So

So L(x) has an absolute minimum when x = 6, and the absolute minimum value of L(x) is

$$L(6) \approx 244.141$$

But L(x) was the square of the length of the ladder, so the length of the shortest ladder that will work is

$$\sqrt{L(6)} = \frac{125}{8} = 15.625\ ft.$$

The angle which this ladder makes with the horizontal is

$$\theta = \tan^{-1}\frac{8}{\frac{125}{8}} = \tan^{-1}\frac{64}{125} \approx 27°.$$

Answer: $15.625\ ft;\ 27°$.