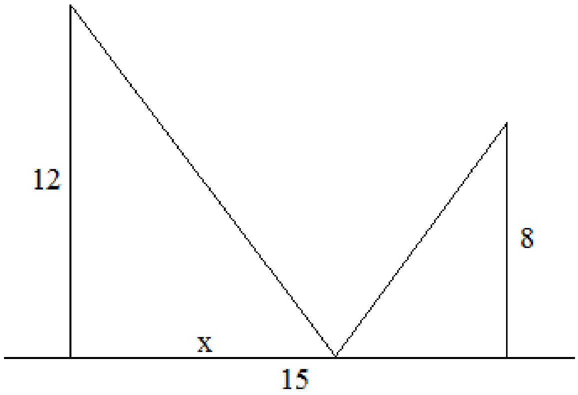
Answer on Question #63948 – Math – Geometry

Question

Two posts, one 8 feet high and the other 12 feet high, stand 15 feet apart. They are to be stayed by wires attached to a single stake at ground level, the wires running to the top of the posts. Where the stake should be placed, to use the least amount of wire?

Solution



Let x be the distance from the 12 foot pole where the wire touches the ground. The length of the wire connecting the 12 foot pole to the ground is $\sqrt{x^2+12^2}$ according to the Pythagorean Theorem. Since the two poles are 15 feet apart, the wire connecting the 8 foot pole will touch the ground 15$-x$ feet from the pole.

The length of this wire is $\sqrt{(15-x)^2+8^2}$.

We have to minimize

$$L(x)=\sqrt{x^2+12^2}+\sqrt{(15-x)^2+8^2}.$$

Compute the first derivative

$$L'(x)=\left(\sqrt{x^2+12^2}+\sqrt{(15-x)^2+8^2}\right)'=\frac{x}{\sqrt{x^2+12^2}}-\frac{15-x}{\sqrt{(15-x)^2+8^2}}.$$

$L'(x)$ is defined for $x\in R$, but we consider $0\le x<+15$.

$L'(x)=0 \Rightarrow \dfrac{x}{\sqrt{x^2+12^2}}-\dfrac{15-x}{\sqrt{(15-x)^2+8^2}}=0 \Rightarrow$

$\Rightarrow \dfrac{x}{\sqrt{x^2+12^2}}=\dfrac{15-x}{\sqrt{(15-x)^2+8^2}};$

$x\sqrt{(15-x)^2+8^2}=(15-x)\sqrt{x^2+12^2}.$

In order to solve for x it is easiest to square both sides and simplify:

$x^2((15-x)^2+8^2)=(15-x)^2(x^2+12^2);$

$x^2(15-x)^2+64x^2=(15-x)^2x^2+144(15-x)^2;$

subtract $x^2(15-x)^2$ from the left-hand and the right-hand sides:

$64x^2=144(15-x)^2;$

divide by 16 through:

$4x^2=9(15-x)^2;$

$4x^2=9(225-30x+x^2);$

$5x^2-270x+2025=0;$

divide by 5:

$x^2 - 54x + 405 = 0;$
$(x-9)(x-45) = 0, 0 \le x \le 15.$
We exclude $x = 45$. Therefore $x = 9$.
Using the First Derivative Test
If we choose $x = 0 \in [0, 9)$, then

$$L'(0) = \frac{0}{\sqrt{0^2 + 12^2}} - \frac{15-0}{\sqrt{(15-0)^2 + 8^2}} = -\frac{15}{17} < 0.$$

If we choose $x = 15 \in (9, 15]$, then

$$L'(15) = \frac{15}{\sqrt{15^2 + 12^2}} - \frac{15-15}{\sqrt{(15-15)^2 + 8^2}} = \frac{15}{\sqrt{369}} > 0.$$

Since the derivative changes from negative to positive, L(x) will have a minimum when $x = 9$.
Therefore the stake should be placed 9 feet from the 12 foot pole.

Answer: the stake should be placed 9 feet from the 12 foot pole.