Task 1. *Find the equation for the circle of curvature of the curve* $r(t) = \sqrt{2}t\hat{\imath} + t^2\hat{\jmath}$ *at the point* $(\sqrt{8}, 4)$ *(this curve parameterizes the curve* $y = \frac{1}{2}x^2$ *in the xy-plane). Along with the problem please explain*

1. *What value of t corresponds to the indicated coordinates?*
2. *What elements will you need for the equation of the circle?*
3. *What values and vectors will aide you in finding the elements of the circle?*
4. *How will you use these values and vectors to obtain the necessary elements?*

Solution. (1) Find the value of the parameter t, corresponding to the point $(\sqrt{8}, 4)$. This value must satisfy the system of equations:

$$\begin{cases} \sqrt{2}t = \sqrt{8} \\ t^2 = 4 \end{cases}.$$

It follows from the first equation that $t = \frac{\sqrt{8}}{\sqrt{2}} = \frac{2\sqrt{2}}{\sqrt{2}} = 2$. Since $2^2 = 4$, the second equation is also satisfied. Thus, $t = 2$.

(2) To write down the equation of the circle, one needs to know the coordinates of the center of this circle and its radius.

(3) To find the center and radius of the circle of curvature, evaluate the velocity and the acceleration vectors of the curve at the point $(\sqrt{8}, 4)$.

$$\begin{aligned} r'(t) &= \sqrt{2}\hat{\imath} + 2t\hat{\jmath}, \\ r''(t) &= 2\hat{\jmath}. \end{aligned}$$

Since $(\sqrt{8}, 4)$ corresponds to $t = 2$, the velocity at this point is $r'(2) = \sqrt{2}\hat{\imath} + 4\hat{\jmath}$. The acceleration $r''(t)$ is constant and equals $2\hat{\jmath}$. We also need the normal vector of the curve at $(\sqrt{8}, 4)$:

$$n(t) = -y'(t)\hat{\imath} + x'(t)\hat{\jmath},$$

where $x'(t)\hat{\imath} + y'(t)\hat{\jmath}$ is the velocity vector. Thus, $n(2) = -4\hat{\imath} + \sqrt{2}\hat{\jmath}$.

(4) The radius R of the circle of curvature equals $\frac{1}{|k|}$, where

$$k = \frac{\det(r'(2), r''(2))}{\|r'(2)\|^3}.$$

Here

$$\det(r'(2), r''(2)) = \begin{vmatrix} \sqrt{2} & 0 \\ 4 & 2 \end{vmatrix} = \sqrt{2} \cdot 2 - 4 \cdot 0 = 2\sqrt{2},$$

$$\|r'(2)\| = \|\sqrt{2}\hat{\imath} + 4\hat{\jmath}\| = \sqrt{\left(\sqrt{2}\right)^2 + 4^2} = \sqrt{2+16} = \sqrt{18} = 3\sqrt{2}.$$

Thus,

$$k = \frac{2\sqrt{2}}{\left(3\sqrt{2}\right)^3} = \frac{2\sqrt{2}}{27\cdot 2\sqrt{2}} = \frac{1}{27},$$

hence $R = 27$.

The position vector $r_C = x_C\hat{\imath} + y_C\hat{\jmath}$ of the center of the circle of curvature can be found by the formula

$$r_C(t) = r(t) + R\frac{n(t)}{\|n(t)\|} = r(t) + R\frac{n(t)}{\|r'(t)\|},$$

since $\|n(t)\| = \|r'(t)\|$. Therefore,

$$\begin{aligned} r_C(2) &= \sqrt{8}\hat{\imath} + 4\hat{\jmath} + 27\cdot\frac{1}{3\sqrt{2}}\cdot(-4\hat{\imath} + \sqrt{2}\hat{\jmath}) \\ &= \sqrt{8}\hat{\imath} + 4\hat{\jmath} + \frac{9}{\sqrt{2}}\cdot(-4\hat{\imath} + \sqrt{2}\hat{\jmath}) \\ &= -16\sqrt{2}\hat{\imath} + 13\hat{\jmath}. \end{aligned}$$

Finally, the equation of the circle is $(x - x_C)^2 + (y - y_C)^2 = R^2$, i.e. $(x + 16\sqrt{2})^2 + (y - 13)^2 = 27^2$.

Answer: $(x + 16\sqrt{2})^2 + (y - 13)^2 = 729$. □