Answer on Question #63787 – Math – Algorithms | Quantitative Methods

Question

Are the following TRUE or FALSE. Explain

n2 = O(n2)

n3 = O(n2)

n log n = O(n2)

n2 = O(n log2 n)

Solution

a) $n^2 = O(n^2)$

TRUE. We can see this if we set *c* = 1, then $n^2 \leq cn^2 = n^2$ for all $n \geq 1$.

Thus, the definition of big-O holds for *c* = 1 and $n_0 = 1$.

b) $n^3 = O(n^2)$

FALSE. For it to be true, we would need that there exist positive constants *c* and n_0 such that $n^3 \leq cn^2$ for all $n \geq n_0$. By dividing both sides by n^2, we see that "$n^3 \leq cn^2$ for all $n \geq n_0$" is true if and only if "n ≤ *c* for all $n \geq n_0$" is true, but clearly there are no constants c and n_0 for which the last statement is true.

c) $n\ log\ n = O(n^2)$

TRUE. Because $\ln(1+x) \leq x$ for all $x > -1$, hence $\ln(1+x) \leq (x+1) - 1 < 2(x+1)$.

Therefore, $log\ n = O(n)$. There exist positive constants *c* and n_0 such that $log\ n \leq cn$ for all

$n \geq n_0$. Multiplying both sides by *n* gives $n\ log\ n \leq cn^2$ for all $n \geq n_0$ for the same positive constants *c* and n_0, so $n\ log\ n = O(n^2)$.

d) $n^2 = O(n \log^2 n)$

FALSE. First, note that $n^2 = O(n \log^2 n)$ if and only if there exist positive constants c and n_0 such that $n^2 \leq cn \log^2 n$ for all $n \geq n_0$, which holds if and only if

$$\frac{n^2}{n \log^2 n} \text{ for all } n \geq n_0 \ (1)$$

By cancelling out the *n* from the numerator and the denominator, we can rewrite expression in (1) as

$$\frac{n}{\log^2 n} \leq c \text{ for all } n \geq n_0.$$

Writing $n = n^{\frac{1}{2}} n^{\frac{1}{2}}$, we see that the last statement is true if and only if

$$\left[\frac{n^{\frac{1}{2}}}{log\, n}\right]^2 \leq c \text{ for all } n \geq n_0.$$

Because we know that $log\, n = o(n^{\frac{1}{2}})$ (in other words, $\lim_{n\to\infty} \frac{\log n}{\sqrt{n}} = 0$), we have that

$$\frac{n^{\frac{1}{2}}}{log\, n} \to \infty \text{ as } n \to \infty.$$

So $\left[\frac{n^{\frac{1}{2}}}{log\, n}\right]^2 \leq c$ cannot be true for all $n \geq n_0$ and for a constant *c*.

Answer: TRUE, FALSE, TRUE, FALSE.