Problem #6345 A normed space which is isometric to a Banach space is itself a Banach space. Prove.

Solution Let $(X, \|\cdot\|_X)$ be a Banach space, $(Y, \|\cdot\|_Y)$ is a normed linear space, which is isometric to X, that is exists $T\colon X \to Y$, which is bijective, linear and "preserves' the distance that is $\|Tx\|_Y = \|x\|_X$. We are to prove that every fundamental sequence $\{y_n\}_n \subset Y$ converges in $Y.T$ is bijective, hence there exists the sequence $\{x_n\}_n \subset X$, such that $y_n = Tx_n$. From isometric we obtain $\|y_n - y_m\|_Y = \|Tx_n - Tx_m\|_Y = \|x_n - x_m\|_X$, thus $\{x_n\}_n$ is fundamental in X, and due to X is Banach space, it converges in X. Denote by $x = \lim x_n$, what is equivalent to $\|x - x_n\|_X \to 0$, $n \to \infty$. Denote by $y = Tx$, then $\|y - y_n\|_Y = \|Tx - Tx_n\|_Y = \|x - x_n\|_X \to 0$, $n \to \infty$. Hence $\{y_n\}_n$ converges to y. To sum it over, we proved that every fundamental sequence in Y converges to some element in Y. We are done.

Answer