Task 1. *Let $S, T : L^2(0,\infty) \to L^2(0,\infty)$ be given by $(Sf)(t) = f(2t), (Tf)(t) = f(t/2)$, f is an element of $L^2(0,\infty)$. You are given that S and T are linear mappings $L^2(0,\infty) \to L^2(0,\infty)$, and you need not prove this fact.*

(i) Calculate $\|S\|$.

(ii) Show that $T^* = 2S$. Determine S^*.

Solution. (i) Take an arbitrary $f \in L^2(0,\infty)$ and consider

$$\|Sf\| = \left(\int_0^\infty |(Sf)(t)|^2 dt\right)^{\frac{1}{2}} = \left(\int_0^\infty |f(2t)|^2 dt\right)^{\frac{1}{2}}.$$

Make the substitution $s = 2t$. Then $t = \frac{s}{2}$ and $dt = \frac{ds}{2}$. If t changes from 0 to ∞, then s also changes from 0 to ∞. This means that after the substitution we obtain

$$\left(\int_0^\infty |f(s)|^2 \frac{ds}{2}\right)^{\frac{1}{2}} = \frac{1}{\sqrt{2}}\left(\int_0^\infty |f(s)|^2 ds\right)^{\frac{1}{2}} = \frac{1}{\sqrt{2}}\|f\|.$$

Thus, $\|Sf\| = \frac{1}{\sqrt{2}}\|f\|$, hence $\|S\| = \frac{1}{\sqrt{2}}$.

(ii) To show that $T^* = 2S$ it is necessary and sufficient to make sure that $(Tf, g) = (f, 2Sg)$ for all $f, g \in L^2(0,\infty)$, where $(\cdot,\cdot)$ denotes the scalar product in the Hilbert space $L^2(0,\infty)$. Indeed,

$$(Tf, g) = \int_0^\infty (Tf)(t)\overline{g(t)}dt = \int_0^\infty f(t/2)\overline{g(t)}dt = \int_0^\infty f(s)\overline{g(2s)}2ds = (f, 2Sg).$$

Here we use the substitution $t = 2s$.

To determine S^* note that $S = \frac{1}{2}T^*$, therefore $S^* = \frac{1}{2}T^{**} = \frac{1}{2}T$.

Answer

(i) $\frac{1}{\sqrt{2}}$.

(ii) $S^* = \frac{1}{2}T$.

□