Answer on Question #62107 – Math – Statistics and Probability

Question

Let X have a standard gamma distribution with α = 7.
Compute P(X<4 or X>6)
0.912
0.625
0.812
0.713

Solution

For $\alpha > 0$ the gamma function is defined as follows:

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx.$$

For integer n: $\Gamma(n) = (n-1)!$

If X is a continuous random variable, then X is said to have a gamma distribution if the pdf of X is

$$f(x) = \frac{\left(\frac{x-\mu}{\beta}\right)^{\alpha-1} \cdot \exp\left(-\frac{x-\mu}{\beta}\right)}{\beta\Gamma(\alpha)}, x \geq \mu;\ \alpha, \beta > 0,$$
$$f(x) = 0, otherwise;$$

where α is the shape parameter, μ is the location parameter, β is the scale parameter, and Γ is the gamma function.
If $\beta = 1$ and $\mu = 0$ then we have the *standard gamma distribution*.

$$f(x) = \frac{x^{\alpha-1} e^{-x}}{\Gamma(\alpha)}, x \geq 0;\ \alpha > 0$$

When X follows the *standard gamma distribution* then its cdf is

$$F(x;\ \alpha) = \int_0^x \frac{y^{\alpha-1} e^{-y}}{\Gamma(\alpha)} dy, x > 0$$

This is also called the incomplete gamma function. The cumulative distribution function of the gamma distribution can be calculated using the function GAMMA.DIST in Microsoft Excel.

$$P(X < 4\ or\ X > 6) = P(X < 4) + P(X > 6) = P(X < 4) + 1 - P(X < 6) =$$
$$= F(4;\ 7) + 1 - F(6; 7) \approx 0.110674 + 0.606303 = 0.716977 \approx$$
$$\approx 0.717.$$

Answer: 0.717.

Question

Let X = the time between two successive arrivals at the drive –up window of a bank. If X has a exponential distribution with h=1 (which is identical to a standard gamma distribution with a=1). Compute the standard deviation of the time between successive arrivals.

2
1
3
4

Solution

The probability density function (pdf) of a exponential distribution is

$$f(x) = \begin{cases} he^{-hx}, x \geq 0, \\ 0, \quad otherwise. \end{cases}$$

In our case $h = 1$ and $f(x) = e^{-x}$, $x \geq 0$.

$$E(X) = \int_{-\infty}^{+\infty} xf(x)dx$$

$$= \int_{0}^{+\infty} xe^{-x}dx$$

$$= -\int_{0}^{+\infty} xde^{-x} = -\left(xe^{-x}|_0^{+\infty} - \int_{0}^{+\infty} e^{-x}dx\right)$$

$$= -(xe^{-x}|_0^{+\infty} + e^{-x}|_0^{+\infty}) = -\lim_{A\to+\infty}(xe^{-x}|_0^A + e^{-x}|_0^A) =$$

$$= -\lim_{A\to+\infty}(Ae^{-A} - 0 + e^{-A} - 1) = 1;$$

$$V(X) = \int_0^{\infty}(x-1)^2 e^{-x}dx = \begin{vmatrix} u = x - 1 & du = dx \\ (x-1)^2 = u^2 & e^{-x} = \frac{1}{e}e^{-u} \end{vmatrix} = \frac{1}{e}\int_{-1}^{\infty} u^2e^{-u}du;$$

$$\int u^2e^{-u}du = -\int u^2d(e^{-u}) = -u^2e^{-u} + 2\int ue^{-u}du = -u^2e^{-u} -$$

$$-2\int ud(e^{-u}) = -u^2e^{-u} - 2ue^{-u} + 2\int e^{-u}du = -u^2e^{-u} - 2ue^{-u} - 2e^{-u}$$

$$V(x) = \frac{1}{e}\cdot\left[-u^2e^{-u} - 2ue^{-u} - 2e^{-u}\,\Big|_{-1}^{\infty}\right] = \frac{1}{e}\cdot[-0 - 0 - 0 + e - 2e + 2e] = 1.$$

The standard deviation is

$\sigma = \sqrt{V(X)} = \sqrt{1} = 1.$

Answer: 1.