Answer on Question #62064 – Math – Differential Equations

Question

A paratrooper weighing 80 kg jumps with zero velocity from an airplane at a height of 3000 m. The air resistance encountered by the paratrooper is $R = 15v(t)$ N, where $v(t)$ is the velocity of the paratrooper at time t. Calculate the time required by the paratrooper to land and the velocity at landing.

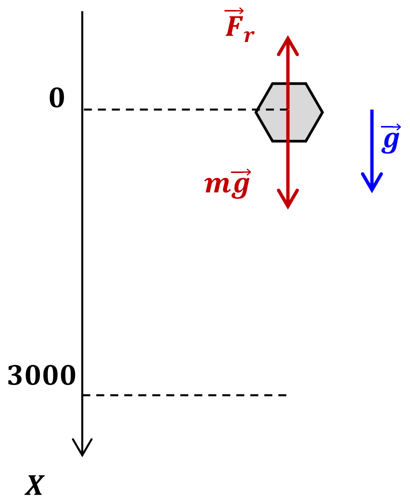
Solution

The paratrooper moves under the influence of gravity and air resistance force. According to Newton's second law:

$$m\vec{a} = m\vec{g} + \vec{F}_r .$$

Choose the direction of axes X as it is shown on the Figure 1 and write down this vector equality in the projections on the coordinate axes X:

$$ma = mg - F_r .$$



Given that

$$a = \frac{dv}{dt} \quad \text{and} \quad F_r = 15v$$

write down the differential equation modeling the situation:

$$m\frac{dv}{dt} = mg - 15v.$$

Separate the variables to obtain an equation connecting two integrals:

$$\frac{dv}{mg-15v} = \frac{dt}{m}.$$

Now integrate both sides of this equation:

$$\int \frac{dv}{mg - 15v} = \int \frac{dt}{m};$$

$$-\frac{1}{15}ln(mg - 15v) = \frac{t}{m} + C_1 .$$

Apply the initial condition to determine the constant C_1.

At time $t = 0$ the velocity $v = 0$ so we have:

$$C_1 = -\frac{1}{15}ln(mg).$$

The solution is:

$$\frac{t}{m} = \frac{1}{15}ln(mg) - -\frac{1}{15}ln(mg - 15v)$$

or finally obtain:

$$t = \frac{m}{15} ln \frac{mg}{mg - 15v}.$$

Express v from the last equation:

$$ln \frac{mg}{mg - 15v} = \frac{15}{m} t;$$

$$\frac{mg}{mg - 15v} = e^{\frac{15}{m}t};$$

$$mg - 15v = mge^{-\frac{15}{m}t};$$

$$v = \frac{mg}{15}\left(1 - e^{-\frac{15}{m}t}\right).$$

Given that

$$v = \frac{dx}{dt}$$

we have:

$$\frac{dx}{dt} = \frac{mg}{15}\left(1 - e^{-\frac{15}{m}t}\right).$$

Separate variables:

$$dx = \frac{mg}{15}\left(1 - e^{-\frac{15}{m}t}\right) dt.$$

Now we integrate both sides and apply the initial condition to get the solution.

Integration the differential equation gives:

$$x = \frac{mg}{15}\left(t + \frac{m}{15} e^{-\frac{15}{m}t}\right) + C_2,$$

where C_2 is a constant.

Applying the initial condition $x = 0$ when $t = 0$ gives:

$$C_2 = -\frac{m^2 g}{225}.$$

Finally:

$$x = \frac{mg}{15}\left(t + \frac{m}{15} e^{-\frac{15}{m}t}\right) - \frac{m^2 g}{225}.$$

At the moment of paratrooper's landing $x = 3000$ so we can calculate the time required by the paratrooper to land:

$$\frac{mgt}{15} + \frac{m^2 g}{225} e^{-\frac{15}{m}t} - \frac{m^2 g}{225} = 3000.$$

Exponential factor $e^{-\frac{15}{m}t}$ rapidly decreases to zero with increasing time (Fig.2) and term $\frac{m^2 g}{225} e^{-\frac{15}{m}t}$ tends to zero.

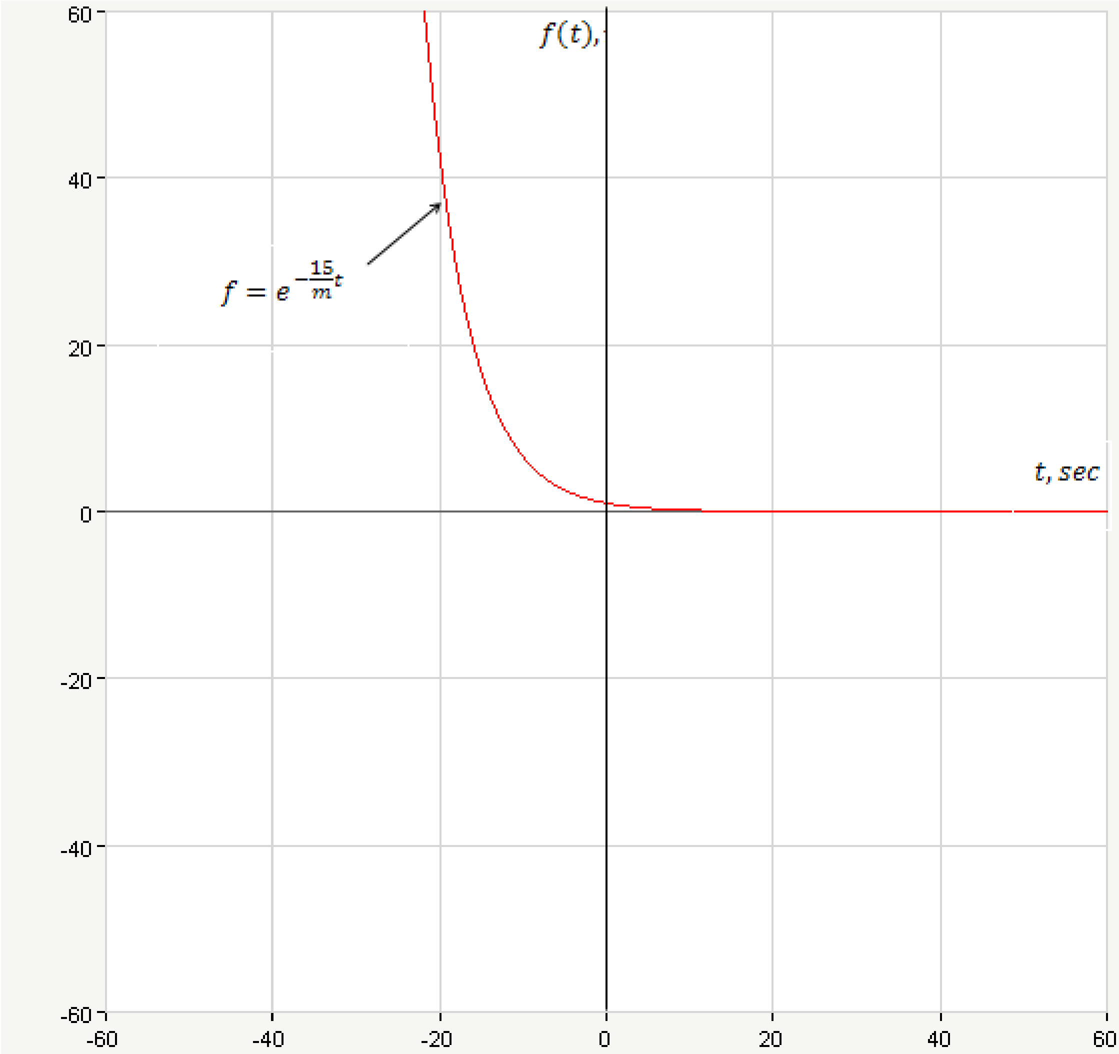


Fig. 2

So we have:

$$\frac{mgt}{15} - \frac{m^2 g}{225} = 3000.$$

The time required to land is

$$t_l = \frac{3000 - \frac{m^2 g}{225}}{\frac{mg}{15}} = \frac{45000}{mg} - \frac{m}{15};$$

$$t_l = \frac{45000}{784} - \frac{80}{15} = 57.4 - 5.3 = 54.4 \text{ s}.$$

The velocity at landing is

$$v_l = \frac{mg}{15};$$

$$v_l = \frac{80 \cdot 9.8}{15} = \frac{784}{15} = 52 \frac{m}{s}.$$

Answer: 54.4 s; 52 $\frac{m}{s}$.