Answer on Question #61225 - Math – Algebra

Question

Find the coefficient of b^{16} in the binomial expansion of $(2b^3 - 1/(4b^2))^{12}$

Solution

Using the binomial theorem (binomial expansion)

$$(x+y)^{12} = \binom{12}{0}x^{12}y^0 + \binom{12}{1}x^{11}y^1 + \binom{12}{2}x^{10}y^2 + \binom{12}{3}x^9y^3 + \binom{12}{4}x^8y^4 + \binom{12}{5}x^7y^5 + \binom{12}{6}x^6y^6 + \binom{12}{7}x^5y^7 + \binom{12}{8}x^4y^8 + + \binom{12}{9}x^3y^9 + \binom{12}{10}x^2y^{10} + \binom{12}{11}xy^{11} + \binom{12}{12}x^0y^{12} =$$

$$= \sum_{k=0}^{12}\binom{12}{k}x^{12-k}y^k = \sum_{l=0}^{12}\binom{12}{l}x^l y^{12-l}.$$

Take $x = 2b^3$, $y = -\frac{1}{4b^2}$.

Then

$$\left(2b^3 - \frac{1}{4b^2}\right)^{12} = (2b^3 - 2^{-2}b^{-2})^{12} = \sum_{l=0}^{12}\binom{12}{l}(2b^3)^l(2^{-2}b^{-2})^{12-l} =$$

$$= \sum_{l=0}^{12}\binom{12}{l}2^l b^{3l}2^{-24+2l}b^{-24+2l} = \sum_{l=0}^{12}\binom{12}{l}2^{3l-24}b^{5l-24}.$$

If $b^{5l-24} = b^{16}$, then $5l - 24 = 16$, hence $5l = 40$, that is, $l = 8$ and this value is in range from 0 to 12.

The coefficient of b^{16} in the binomial expansion of $\left(2b^3 - \frac{1}{4b^2}\right)^{12}$ is

$$\binom{12}{l}2^{3l-25} = \binom{12}{8}2^{3\cdot8-24} = \binom{12}{8}2^0 = \frac{12!}{8!\,(12-8)!} = \frac{12\cdot11\cdot10\cdot9\cdot8!}{8!\,4!} =$$

$$= \frac{12\cdot11\cdot10\cdot9}{4\cdot3\cdot2} = 495.$$

Answer: 495.