Answer on Question #61150 – Math – Algebra

Question

Solve the equation

$$z^4 = -2\sqrt{3} - 2i \qquad (1)$$

Sketch the solution in the complex plane.

Solution

The absolute value (the modulus) of the complex number $-2\sqrt{3} - 2i$ is

$$\rho(z^4) = \sqrt{\left(-2\sqrt{3}\right)^2 + (-2)^2} = 4$$

The argument of the complex number $-2\sqrt{3} - 2i$ is

$\theta(z^4) = -\pi + \tan^{-1}\left(\frac{-2}{-2\sqrt{3}}\right) = -\pi + \frac{\pi}{6} = -\frac{5\pi}{6}$ or $2\pi - \frac{5\pi}{6} = \frac{7\pi}{6}$.

The absolute value (the modulus) of the complex number $z = \sqrt[4]{-2\sqrt{3} - 2i}$ is

$$\rho(z) = \sqrt[4]{4} = \sqrt{2} \approx 1.41 \qquad (2)$$

The argument of the complex number $z = \sqrt[4]{-2\sqrt{3} - 2i}$ is

$$\theta(z) = \frac{\frac{7\pi}{6} + 2\pi k}{4}, k = 0, 1, 2, 3. \qquad (3)$$

Solutions to the equation (1) are given by

$$z = \rho(z) \cdot (\cos\theta(z) + i\sin\theta(z)),$$

where $\rho(z)$ and $\theta(z)$ are defined by means of formulae (2) and (3).

If $k = 0$, then $\theta(z) = \theta_1 = \frac{\frac{7\pi}{6}}{4} = \frac{7\pi}{24} \approx 52.5^\circ$,

$\cos\frac{7\pi}{24} = \frac{1}{2}\sqrt{2 - \sqrt{2 - \sqrt{3}}} \approx 0.61$ (see http://mathworld.wolfram.com/TrigonometryAnglesPi24.html),

$\sin\frac{7\pi}{24} = \frac{1}{2}\sqrt{2 + \sqrt{2 - \sqrt{3}}} \approx 0.79$ (see http://mathworld.wolfram.com/TrigonometryAnglesPi24.html),

and a solution will be

$$z = z_1 = \rho(z) \cdot (\cos\theta_1 + i\sin\theta_1) = \sqrt{2}\left(\cos\frac{7\pi}{24} + i\sin\frac{7\pi}{24}\right) \approx 0.86 + 1.12i.$$

If $k = 1$, then $\theta(z) = \theta_2 = \frac{\frac{7\pi}{6} + 2\pi}{4} = \frac{19\pi}{24} \approx 142.5^\circ$,

$$\cos\frac{19\pi}{24} = \cos\left(\pi - \frac{5\pi}{24}\right) = -\cos\frac{5\pi}{24} = -\frac{1}{2}\sqrt{2 + \sqrt{2 - \sqrt{3}}} \approx -0.79$$

(see http://mathworld.wolfram.com/TrigonometryAnglesPi24.html),

$$\sin\frac{19\pi}{24} = \sin\left(\pi - \frac{5\pi}{24}\right) = \sin\frac{5\pi}{24} = \frac{1}{2}\sqrt{2-\sqrt{2-\sqrt{3}}} \approx 0.61$$

(see http://mathworld.wolfram.com/TrigonometryAnglesPi24.html),

and a solution will be

$$z = z_2 = \rho(z) \cdot (\cos\theta_2 + i\sin\theta_2) = \sqrt{2}\left(\cos\frac{19\pi}{24} + i\sin\,\frac{19\pi}{24}\right) \approx -1.12 + 0.86i.$$

If $k = 2$, then $\theta(z) = \theta_3 = \frac{\frac{7\pi}{6}+4\pi}{4} = \frac{31\pi}{24} \approx 232.5^\circ$,

$$\cos\frac{31\pi}{24} = \cos\left(\pi + \frac{7\pi}{24}\right) = -\cos\frac{7\pi}{24} = -\frac{1}{2}\sqrt{2-\sqrt{2-\sqrt{3}}} \approx -0.61$$

(see http://mathworld.wolfram.com/TrigonometryAnglesPi24.html),

$$\sin\frac{31\pi}{24} = \sin\left(\pi + \frac{7\pi}{24}\right) = -\sin\frac{7\pi}{24} = -\frac{1}{2}\sqrt{2+\sqrt{2-\sqrt{3}}} \approx -0.79$$

(see http://mathworld.wolfram.com/TrigonometryAnglesPi24.html),

and a solution will be

$$z = z_3 = \rho(z) \cdot (\cos\theta_3 + i\sin\theta_3) = \sqrt{2}\left(\cos\frac{31\pi}{24} + i\sin\,\frac{31\pi}{24}\right) \approx -0.86 - 1.12i$$

If $k = 3$, then $\theta(z) = \theta_4 = \frac{\frac{7\pi}{6}+6\pi}{4} = \frac{43\pi}{24} \approx 322.5^\circ$,

$$\cos\frac{43\pi}{24} = \cos\left(\frac{48\pi - 5\pi}{24}\right) = \cos\left(2\pi - \frac{5\pi}{24}\right) = \cos\left(-\frac{5\pi}{24}\right) = \cos\frac{5\pi}{24} = \frac{1}{2}\sqrt{2+\sqrt{2-\sqrt{3}}} \approx 0.79$$

(see http://mathworld.wolfram.com/TrigonometryAnglesPi24.html),

$$\sin\frac{43\pi}{24} = \sin\left(\frac{48\pi - 5\pi}{24}\right) = \sin\left(2\pi - \frac{5\pi}{24}\right) = \sin\left(-\frac{5\pi}{24}\right) = -\sin\frac{5\pi}{24} = -\frac{1}{2}\sqrt{2-\sqrt{2-\sqrt{3}}} \approx -0.61$$

(see http://mathworld.wolfram.com/TrigonometryAnglesPi24.html),

and a solution will be

$$z = z_4 = \rho(z) \cdot (\cos\theta_4 + i\sin\theta_4) = \sqrt{2}\left(\cos\frac{43\pi}{24} + i\sin\,\frac{43\pi}{24}\right) \approx 1.12 - 0.86i$$

Table 1. Arguments of the solutions

$\theta(z)$
$\frac{7\pi}{24}$
$\frac{19\pi}{24}$
$\frac{31\pi}{24}$
$\frac{43\pi}{24}$

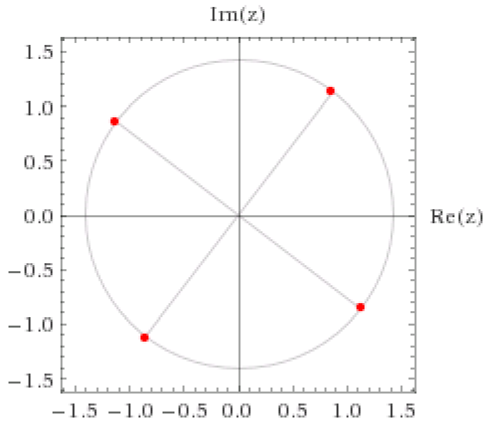

Figure 1. The solutions to the equation $z^4 = -2\sqrt{3} - 2i$ in the complex plane

Now show how to deduce auxiliary formulae, for example,

$\cos\frac{5\pi}{24} = \frac{1}{2}\sqrt{2+\sqrt{2-\sqrt{3}}}$, $\sin\frac{5\pi}{24} = \frac{1}{2}\sqrt{2-\sqrt{2-\sqrt{3}}}$.

It is known that $\cos 30^\circ = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$, $\sin 30^\circ = \sin\frac{\pi}{6} = \frac{1}{2}$.

Using half-angle formulae

$$\sin\frac{2\pi}{24} = \sin 15^\circ = \sin\frac{30^\circ}{2} = \sqrt{\frac{1-\cos 30^\circ}{2}} = \sqrt{\frac{1-\frac{\sqrt{3}}{2}}{2}} = \frac{1}{2}\sqrt{2-\sqrt{3}},$$

$$\cos\frac{2\pi}{24} = \cos 15^\circ = \cos\frac{30^\circ}{2} = \sqrt{\frac{1+\cos 30^\circ}{2}} = \sqrt{\frac{1+\frac{\sqrt{3}}{2}}{2}} = \frac{1}{2}\sqrt{2+\sqrt{3}}.$$

Using reduction formulae

$$\cos 75^\circ = \sin(90^\circ - 75^\circ) = \sin 15^\circ = \frac{1}{2}\sqrt{2-\sqrt{3}},$$

$$\sin 75^\circ = \sin(90^\circ - 15^\circ) = \cos 15^\circ = \frac{1}{2}\sqrt{2+\sqrt{3}}$$

Using half-angle formulae finally obtain

$$\sin 37.5^\circ = \sin\frac{5\pi}{24} = \sin\left(\frac{5\pi/12}{2}\right) = \sin\left(\frac{75^\circ}{2}\right) = \sqrt{\frac{1-\cos 75^\circ}{2}} = \sqrt{\frac{1-\frac{1}{2}\sqrt{2-\sqrt{3}}}{2}} = \frac{1}{2}\sqrt{2-\sqrt{2-\sqrt{3}}},$$

$$\cos 37.5^\circ = \cos\frac{5\pi}{24} = \cos\left(\frac{5\pi/12}{2}\right) = \cos\left(\frac{75^\circ}{2}\right) = \sqrt{\frac{1+\cos 75^\circ}{2}} = \sqrt{\frac{1+\frac{1}{2}\sqrt{2-\sqrt{3}}}{2}} = \frac{1}{2}\sqrt{2+\sqrt{2-\sqrt{3}}}.$$