Answer on Question #60921 – Math – Algebra

Question

Find in its simples form the quadratic equation with the pair of solution 1/2 ± 3i

Solution

Method 1

If $x_1 = \frac{1}{2} - 3i$, $x_2 = \frac{1}{2} + 3i$ are the roots of the quadratic equation $a_2 x^2 + a_1 x + a_0 = 0$, then

$$a_2 x^2 + a_1 x + a_0 = a_2 (x - x_1)(x - x_2) = a_2 \left(x - \frac{1}{2} + 3i\right)\left(x - \frac{1}{2} - 3i\right) =$$

$$= a_2 \left(\left(x - \frac{1}{2}\right) + 3i\right) \cdot \left(\left(x - \frac{1}{2}\right) - 3i\right) = a_2 \left(\left(x - \frac{1}{2}\right)^2 - (3i)^2\right) =$$

$$a_2 \left(x^2 - 2x \cdot \frac{1}{2} + \frac{1}{4} - 9i^2\right) == a_2 \left(x^2 - 2x \cdot \frac{1}{2} + \frac{1}{4} + 9\right) = a_2 \left(x^2 - x + \frac{37}{4}\right) = 0.$$

Formulae $(d - e)(d + e) = d^2 - e^2$, $i^2 = -1$ were used in the previous chain of expressions.

If we put $a_2 = 4$, then we obtain

$$a_2 x^2 + a_1 x + a_0 = a_2 \left(x^2 - x + \frac{37}{4}\right) = 4\left(x^2 - x + \frac{37}{4}\right) = 4x^2 - 4x + 37 = 0.$$

Therefore, $4x^2 - 4x + 37 = 0$ is the simplest form of the quadratic equation with the pair of solutions $\frac{1}{2} \pm 3i$.

Method 2

By Vieta's formulae, for the quadratic polynomial $P(x) = ax^2 + bx + c$, roots x_1, x_2 of the equation $P(x) = 0$ satisfy

$$x_1 + x_2 = -\frac{b}{a},\ x_1 x_2 = \frac{c}{a}.$$

Hence

$$\begin{cases} x_1 + x_2 = \frac{1}{2} - 3i + \frac{1}{2} + 3i = \frac{1}{2} + \frac{1}{2} = 1 = -\frac{b}{a}, \\ x_1 x_2 = \left(\frac{1}{2} - 3i\right)\left(\frac{1}{2} + 3i\right) = \frac{1}{4} - 9i^2 = \frac{1}{4} + 9 = \frac{37}{4} = \frac{c}{a}, \end{cases}$$

Formulae $(d-e)(d+e) = d^2 - e^2$, $i^2 = -1$ were used in the second line of the system.

So

$$\begin{cases} -\frac{b}{a} = 1, \\ \frac{c}{a} = \frac{37}{4}. \end{cases}$$

If we put $a = 4$, then we obtain $-b = 4$, $c = 37$, hence $b = -4$,

$P(x) = ax^2 + bx + c = 4x^2 - 4x + 37.$

Thus, $4x^2 - 4x + 37 = 0$ is the simplest form of the quadratic equation with the pair of solutions $\frac{1}{2} \pm 3i$.

Method 3

If $x_1 = \frac{1}{2} - 3i$, $x_2 = \frac{1}{2} + 3i$ are the roots of the quadratic equation $a_2x^2 + a_1x + a_0 = 0$, then

$$\begin{cases} a_2x_1^2 + a_1x_1 + a_0 = 0, \\ a_2x_2^2 + a_1x_2 + a_0 = 0, \end{cases}$$

$$\begin{cases} a_2\left(\frac{1}{2} - 3i\right)^2 + a_1\left(\frac{1}{2} - 3i\right) + a_0 = 0, \\ a_2\left(\frac{1}{2} + 3i\right)^2 + a_1\left(\frac{1}{2} + 3i\right) + a_0 = 0, \end{cases}$$

$$\begin{cases} a_2\left(\frac{1}{4} - 3i + 9i^2\right) + a_1\left(\frac{1}{2} - 3i\right) + a_0 = 0, \\ a_2\left(\frac{1}{4} + 3i + 9i^2\right) + a_1\left(\frac{1}{2} + 3i\right) + a_0 = 0, \end{cases}$$

$$\begin{cases} a_2\left(\frac{1}{4} - 3i - 9\right) + a_1\left(\frac{1}{2} - 3i\right) + a_0 = 0, \\ a_2\left(\frac{1}{4} + 3i - 9\right) + a_1\left(\frac{1}{2} + 3i\right) + a_0 = 0, \end{cases}$$

$$\begin{cases}\left(-\frac{35}{4}a_2+\frac{1}{2}a_1+a_0\right)-3i(a_1+a_2)=0+0\cdot i,\\ \left(-\frac{35}{4}a_2+\frac{1}{2}a_1+a_0\right)+3i(a_1+a_2)=0+0\cdot i.\end{cases}$$

The real and imaginary parts in the left-hand and the right-hand sides of both equations are equal.

Then come to the following system of equations:

$$\begin{cases}-\frac{35}{4}a_2+\frac{1}{2}a_1+a_0=0,\\ a_1+a_2=0,\end{cases}$$

$$\begin{cases}-\frac{35}{4}a_2+\frac{1}{2}a_1+a_0=0,\\ a_2=-a_1,\end{cases}$$

$$\begin{cases}-\frac{35}{4}\cdot(-a_1)+\frac{1}{2}a_1+a_0=0,\\ a_2=-a_1,\end{cases}$$

$$\begin{cases}\frac{37}{4}a_1+a_0=0,\\ a_2=-a_1,\end{cases}$$

$$\begin{cases}a_0=-\frac{37}{4}a_1,\\ a_2=-a_1,\end{cases}$$

If we put $a_1=-4$, then we obtain $a_0=-\frac{37}{4}a_1=-\frac{37}{4}\cdot(-4)=37;$

$a_2=-a_1=-(-4)=4,\ a_2x^2+a_1x+a_0=4x^2-4x+37=0.$

Thus, $4x^2-4x+37=0$ is the simplest form of the quadratic equation with the pair of solutions $\frac{1}{2}\pm 3i$.

Method 4

We shall prove that $x_{1,2}=\frac{1}{2}\pm 3i$ are roots of the quadratic equation

$4x^2-4x+37=0.$

Indeed,

$$4\left(\frac{1}{2}+3i\right)^2-4\cdot\left(\frac{1}{2}+3i\right)+37=4\cdot\left(\frac{1}{4}+3i-9\right)-4\cdot\left(\frac{1}{2}+3i\right)+37=$$

$$=1+12i-36-2-12i+37=1-36-2+37=0,$$

$$4\left(\frac{1}{2}-3i\right)^2-4\cdot\left(\frac{1}{2}-3i\right)+37=4\cdot\left(\frac{1}{4}-3i-9\right)-4\cdot\left(\frac{1}{2}-3i\right)+37=$$

$$=1-12i-36-2+12i+37=1-36-2+37=0.$$

Thus, $4x^2-4x+37=0$ is the simplest form of the quadratic equation with the pair of solutions $\frac{1}{2}\pm 3i$.

Answer: $4x^2-4x+37=0.$