Answer on Question #60847 – Math – Algebra

Question

On 1 Jan 2014 a bird sanctuary is home to 25 birds. During Jan 2014, 3 more birds are brought into the sanctuary. During each subsequent month, 2 more birds are brought into the sanctuary than were brought in during the previous month. Assuming that no birds die or leave the sanctuary, work out how many birds are in the sanctuary on 31 Dec 2014.

Solution

On 1 Jan 2014 a bird sanctuary is home to 25 birds.

During January 2014 a bird sanctuary is home to 25 + 3 = 28 birds.

During each subsequent month, 2 more birds are brought into the sanctuary than were brought in during the previous month.

Let $a_1 = 25 + 3 = 28$. Then $a_2 = a_1 + 5 = a_1 + (2 \cdot 1 + 3) = 28 + 5 = 33$,

$a_3 = a_2 + 7 = a_2 + (2 \cdot 2 + 3) = 33 + 7 = 40$,

$a_4 = a_3 + 9 = a_3 + (2 \cdot 3 + 3) = 40 + 9 = 49$.

Therefore the general formula for n^{th} term is

$a_{n+1} = a_n + 2n + 3$, where $a_0 = 25$,

hence

$$a_{n+1} - a_n = 2n + 3. \quad \textbf{(1)}$$

To prove that (1) is valid, we shall use the method of mathematical induction.

If $n = 0$, then $a_1 - a_0 = 3$ is true, because it is given that 3 more birds are brought into the sanctuary during Jan 2014.

Let us assume that formula (1) holds for $n = k$: $a_{k+1} - a_k = 2k + 3$.

Consider the case $n = k + 1$. It is given that during each subsequent month, 2 more birds are brought into the sanctuary than were brought in during the previous month.

If $(2k + 3)$ birds are brought into the sanctuary during $(k+1)^{th}$ month, then

$(2k + 3) + 2 = (2k + 2) + 3 = 2(k + 1) + 3$ birds are brought into the sanctuary during $(k+2)^{th}$ month.

In other words, we proved that $a_{k+2} - a_{k+1} = 2(k + 1) + 3$ holds, hence formula (1) holds for $n = k + 1$.

According to the principle of mathematical induction, formula (1) is valid for any non-negative integer number n.

Method 1

Sequence $b_k = k, k \geq 0$, is arithmetic, where $d = 1$, the formula for the sum gives

$$\sum_{k=0}^{11} k = \sum_{k=1}^{11} k = (1+2+3+4+5+6+7+8+9+10+11) = \frac{1+11}{2} \cdot 11 =$$

$= 6 \cdot 11 = 66.$

Sequence $c_k = 1, k \geq 0$, is constant, the sum is

$$\sum_{k=0}^{11} 1 = 1 \cdot 12 = 12$$

Using formula (1) and summing up all differences $a_{k+1} - a_k$ from $k = 0$ to 11 obtain

$\sum_{k=0}^{11}(a_{k+1} - a_k) = a_1 - a_0 + a_2 - a_1 + a_3 - a_2 + a_4 - a_3 + a_5 - a_4 + a_6 - a_5 +$
$+ a_7 - a_6 + a_8 - a_7 + a_9 - a_8 + a_{10} - a_9 + a_{11} - a_{10} + a_{12} - a_{11} =$

$$= (a_1 - a_1) + (a_2 - a_2) + (a_3 - a_3) + (a_4 - a_4) + (a_5 - a_5) + (a_6 - a_6) + (a_7 - a_7) + (a_8 - a_8) + (a_9 - a_9) + (a_{10} - a_{10}) + (a_{11} - a_{11}) + a_{12} - a_0 =$$

$$= a_{12} - a_0 = \sum_{k=0}^{11}(2k+3) = 2\sum_{k=0}^{11} k + 3\sum_{k=0}^{11} 1 =$$

$= 2 \cdot 66 + 3 \cdot 12 = 132 + 36 = 168$, hence

$a_{12} = a_0 + \sum_{k=0}^{11}(a_{k+1} - a_k) = 25 + 168 = 193.$

Answer: 193.

Method 2

$$a_{n+1} = a_n + 2n + 3, \text{ where } a_0 = 25, \quad (2)$$

is non-homogeneous recurrence equation.

The characteristic equation is $r^{n+1} - r^n = 0$, $r^n(r-1) = 0$, hence $r_1 = 0$, $r_2 = 1$. So $a_n = C_1 0^n + C_2 \cdot 1^n = C_2$.

Let $b_n = cn^2 + dn$ be a particular solution, then

$b_{n+1} = b_n + 2n + 3,$

$c(n+1)^2 + d(n+1) = cn^2 + dn + 2n + 3,$

$$2cn + c + d = 2n + 3,$$

hence $2c = 2, c = 1, c + d = 3, d = 3 - c = 3 - 1 = 2.$

Thus, $b_n = cn^2 + dn = n^2 + 2n$ is a particular solution of non-homogeneous equation.

The general solution of non-homogeneous equation is $a_n = n^2 + 2n + C$, where C is arbitrary real constant.

Using $a_0 = 25$ obtain $a_0 = 0^2 + 2 \cdot 0 + C = C = 25.$

Thus, $a_n = n^2 + 2n + 25$ is the solution of problem (2).

Finally $a_{12} = 12^2 + 2 \cdot 12 + 25 = 144 + 24 + 25 = 144 + 49 = 193.$

Answer: 193.

Method 3

On 1 Jan 2014 a bird sanctuary is home to 25 birds.

During January 2014 a bird sanctuary is home to (25 + 3) = 28 birds.

During each subsequent month, 2 more birds are brought into the sanctuary than were brought in during the previous month.

During February 2014 a bird sanctuary is home to (25+3)+3+2=28 +3+ 2 =28+5= 33 birds.

During March 2014 a bird sanctuary is home to (25+3)+3+2+2=33+5+ 2 =33+7= 40 birds.

During April 2014 a bird sanctuary is home to 40+7+ 2 = 40+9=49 birds.

During May 2014 a bird sanctuary is home to 49+ 9+2 = 49+11=60 birds.

During June 2014 a bird sanctuary is home to 60+11+ 2 =60+13= 73 birds.

During July 2014 a bird sanctuary is home to 73+13+2=73+15=88 birds.

During August 2014 a bird sanctuary is home to 88+15+ 2=88+17 = 105 birds.

During September 2014 a bird sanctuary is home to 105+17+ 2 =105+19= 124 birds.

During October 2014 a bird sanctuary is home to 124+19+2=124+21= 145 birds.

During November 2014 a bird sanctuary is home to 145+21+ 2 =145+23= 168 birds.

During December 2014 a bird sanctuary is home to 168+23+ 2 =168+25= 193 birds.

Answer: 193 birds.