Answer on Question #60840 – Math – Algorithms | Quantitative Methods

Question

b) Set up the Gaussi-Jacobi iteration scheme in matrix form for the linear system of equations

4 3
4 2
4 3
2 3
1 2 3
1 2
− + =
− + − =
− =
x x
x x x
x x

Show that the iteration scheme is convergent. Hence find the rate of convergence of this method.

Solution

We have the system of equations $Ax = b$:

$$\begin{pmatrix} 4 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 3 \end{pmatrix}$$

By Gauss-Jacobi method we split A into:

$$A = D - L - U$$

We have

$$D = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix},\ L = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},\ U = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Then

$$x^{(k+1)} = D^{-1}(L+U)x^{(k)} + D^{-1}b$$

At the first iteration $x^{(0)} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$:

$$x^{(1)} = D^{-1}(L+U)x^{(0)} + D^{-1}b = D^{-1}b = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}^{-1}\begin{pmatrix} 3 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1/4 & 0 & 0 \\ 0 & 1/4 & 0 \\ 0 & 0 & 1/4 \end{pmatrix}\begin{pmatrix} 3 \\ 2 \\ 3 \end{pmatrix}$$

Then

$$x^{(1)} = \begin{pmatrix} 0.75 \\ 0.5 \\ 0.75 \end{pmatrix}$$

At the second iteration

$$x^{(2)} = D^{-1}(L+U)x^{(1)} + D^{-1}b = \begin{pmatrix} 0.25 & 0 & 0 \\ 0 & 0.25 & 0 \\ 0 & 0 & 0.25 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0.75 \\ 0.5 \\ 0.75 \end{pmatrix} + \begin{pmatrix} 0.75 \\ 0.5 \\ 0.75 \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & 0.25 & 0 \\ 0.25 & 0 & 0.25 \\ 0 & 0.25 & 0 \end{pmatrix} \begin{pmatrix} 0.75 \\ 0.5 \\ 0.75 \end{pmatrix} + \begin{pmatrix} 0.75 \\ 0.5 \\ 0.75 \end{pmatrix} = \begin{pmatrix} 0.125 \\ 0.1875 \\ 0.125 \end{pmatrix} + \begin{pmatrix} 0.75 \\ 0.5 \\ 0.75 \end{pmatrix} = \begin{pmatrix} 0.875 \\ 0.6875 \\ 0.875 \end{pmatrix}$$

At the third iteration

$$x^{(3)} = D^{-1}(L+U)x^{(2)} + D^{-1}b = \begin{pmatrix} 0.25 & 0 & 0 \\ 0 & 0.25 & 0 \\ 0 & 0 & 0.25 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0.875 \\ 0.6875 \\ 0.875 \end{pmatrix} + \begin{pmatrix} 0.75 \\ 0.5 \\ 0.75 \end{pmatrix} =$$

$$\begin{pmatrix} 0.21875 \\ 0.34375 \\ 0.21875 \end{pmatrix} + \begin{pmatrix} 0.75 \\ 0.5 \\ 0.75 \end{pmatrix} = \begin{pmatrix} 0.96875 \\ 0.84275 \\ 0.96875 \end{pmatrix}$$

As we see it converges to

$$x = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

The scheme is convergent, because the matrix $A = \begin{pmatrix} 4 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 4 \end{pmatrix}$ is positive-definite by Sylvester's criterion.

Indeed,

$\Delta_1 = 4 > 0$, $\Delta_2 = \begin{vmatrix} 4 & -1 \\ -1 & 4 \end{vmatrix} = 4 \cdot 4 - (-1) \cdot (-1) = 16 - 1 = 15 > 0,$

$$\Delta_3 = \begin{vmatrix} 4 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 4 \end{vmatrix} = 4 \begin{vmatrix} 4 & -1 \\ -1 & 4 \end{vmatrix} + \begin{vmatrix} -1 & -1 \\ 0 & 4 \end{vmatrix} = 4 \cdot 15 - 4 = 56 > 0$$

It follows from

$$x^{(k+1)} = D^{-1}(L+U)x^{(k)} + D^{-1}b$$

that

$x^{(k+1)} = Tx^{(k)} + c$, where $k \geq 0$,

$$T = D^{-1}(L+U) = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}^{-1} \cdot \left(\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \right) = \begin{pmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{4} \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & 1/4 & 0 \\ 1/4 & 0 & 1/4 \\ 0 & 1/4 & 0 \end{pmatrix},$$

$$c = D^{-1}b = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}^{-1} \begin{pmatrix} 3 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1/4 & 0 & 0 \\ 0 & 1/4 & 0 \\ 0 & 0 & 1/4 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 0.75 \\ 0.5 \\ 0.75 \end{pmatrix}.$$

Eigenvalues of matrix T are $\lambda_1 = -\frac{1}{2\sqrt{2}}, \lambda_2 = \frac{1}{2\sqrt{2}}, \lambda_3 = 0$, hence the spectral radius is

$\rho(T) = msx\{|\lambda_1|, |\lambda_2|, |\lambda_3|\} = \frac{1}{2\sqrt{2}} < 1.$

Choose a matrix norm and calculate

$\|T\|_2 = \sqrt{\sum_{i=1}^{3} \sum_{j=1}^{3} |t_{ij}|^2} = \sqrt{\left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^2} = \sqrt{4 \cdot \left(\frac{1}{4}\right)^2} = 2 \cdot \frac{1}{4} = \frac{1}{2} < 1.$

The following error bound hold:

$\|x - x^{(k)}\| \leq \|T\|^k \|x^{(0)} - x\|,$

$\|x - x^{(k)}\| \leq \frac{\|T\|^k}{1-\|T\|} \|x^{(1)} - x^{(0)}\|.$

If $\|T\|^k < \varepsilon$, then $k < \frac{log(\varepsilon)}{log(\|T\|)}$.

Asymptotically the error vector e_k behaves at worst like $(\rho(T))^k$, where $e_k = x - x^{(k)}$,

$e_0 = x - x^{(0)}$.