Answer on Question #59112 – Math – Abstract Algebra

Theorem

Let G be a group with identity element e, and let H be a subset of G. Then H is a subgroup of G if and only if the following conditions hold:

(i) $ab \in H$ for all $a,b \in H$;
(ii) $e \in H$;
(iii) $a^{-1} \in H$ for all $a \in H$.

Question

1) Prove that H={a+ib ЄC, a^2+b^2=1} is a subgroup of C, where C is complex number.

Solution

Check 3 conditions according to Theorem.

i. Consider arbitrarily $h_1, h_2 \in H$: $h_1=a_1+ib_1$, $a_1^2+b_1^2=1$; $h_2=a_2+ib_2$, $a_2^2+b_2^2=1$.
Prove that $h_1h_2 \in H$.
$h_1h_2=(a_1+ib_1)(a_2+ib_2)=(a_1a_2-b_1b_2)+i(b_1a_2+a_1b_2)$;
but $(a_1a_2-b_1b_2)^2+(b_1a_2+a_1b_2)^2= a_1^2\, a_2^2 -2\,a_1\, a_2\, b_1\, b_2 + b_1^2\, b_2^2 + b_1^2\, a_2^2 + 2\,b_1 a_2 a_1 b_2 + a_1^2\, b_2^2 =$
$= a_1^2\,(a_2^2+b_2^2) + b_1^2\,(a_2^2+b_2^2) = (a_2^2+b_2^2)(a_1^2+b_1^2)=1$, then $h_1h_2 \in H$. ;

ii. Prove that for $h \in H$ we have $h^{-1} \in H$:

$$h^{-1}=(a+ib)^{-1}= \frac{1}{a+ib} = \frac{a-ib}{(a+ib)(a-ib)} = \frac{a-ib}{a^2+b^2} = \frac{a}{a^2+b^2} + i\frac{-b}{a^2+b^2};$$

but $\left(\frac{a}{a^2+b^2}\right)^2 + \left(\frac{-b}{a^2+b^2}\right)^2 = \frac{a^2}{(a^2+b^2)^2} + \frac{b^2}{(a^2+b^2)^2} = \frac{a^2+b^2}{(a^2+b^2)^2} = \frac{1}{1} = 1$, so $h^{-1} \in H$:

iii. Identity element $e=1=1+0i \in H$.

From conditions i) and ii), iii) of Theorem it follows that H is a subgroup of C.

Note that associativity follows from the associativity of complex numbers multiplication.

Question

2) H be set of real number a+b√2 where a,b Є **Q**. Show that H be a subgroup of non-zero real no. under multiplication.

Solution

Check 3 conditions of Theorem:
i. Consider arbitrarily $h_1, h_2 \in H$: $h_1 = a_1+b_1\sqrt{2}$ $h_2 = a_2+b_2\sqrt{2}$; Prove that $h_1h_2 \in H$.

$h_1h_2 = (a_1+b_1\sqrt{2})(a_2+b_2\sqrt{2}) = (a_1 a_2 + 2 b_1 b_2) + \sqrt{2}(a_1 b_2 + b_1 a_2)$;
a,b ∈ **Q** then $c=(a_1 a_2 + 2 b_1 b_2)$ ∈ **Q** and $d=(a_1 b_2 + b_1 a_2)$ ∈ **Q**, so $h_1h_2 = c + d\sqrt{2} \in H$.

ii. $h^{-1}=(a+b\sqrt{2})^{-1} = \frac{1}{a+b\sqrt{2}} = \frac{a-b\sqrt{2}}{(a+b\sqrt{2})(a-b\sqrt{2})} = \frac{a-b\sqrt{2}}{a^2-2b^2} = \frac{a}{a^2-2b^2} + \frac{-b}{a^2-2b^2}\sqrt{2}$;
a,b ∈ **Q**, $s=\frac{a}{a^2-2b^2}$; $t=\frac{-b}{a^2-2b^2}$, s,t ∈ **Q**; $h^{-1}=s+t\sqrt{2} \in H$;

iii. Identity element $e=1=1+0\sqrt{2} \in H$

From conditions i) and ii) , iii) of Theorem it follows that H is a subgroup of non-zero real no. under multiplication.

Note that associativity follows from the associativity of multiplication of the real numbers.