Answer on Question #58425 – Math – Vector Calculus

Question

1. $\varphi = 2xz^4 - x^2y$ Find $|\nabla\vec{\varphi}|$ at some point.

Solution

$\nabla\vec{\varphi}=\frac{\partial\varphi}{\partial x}\vec{\imath}+\frac{\partial\varphi}{\partial y}\vec{\jmath}+\frac{\partial\varphi}{\partial z}\vec{k} = (2z^4 - 2xy)\vec{\imath} - x^2\vec{\jmath} + 8xz^3\vec{k}$

$$|\nabla\vec{\varphi}| = \sqrt{(2z^4 - 2xy)^2 + x^4 + 64x^2z^6}$$

Point (x, y, z) was not specified.

Question

2. $\varphi = 3x^2y - y^3z^2$ Find $\nabla\vec{\varphi}$ at point (1;-2;-1)

Solution

$\nabla\vec{\varphi} = \frac{\partial\varphi}{\partial x}\vec{\imath}+\frac{\partial\varphi}{\partial y}\vec{\jmath}+\frac{\partial\varphi}{\partial z}\vec{k} = 6xy\vec{\imath} + (3x^2 - 3y^2z^2)\vec{\jmath} - 2zy^3\vec{k}$

Then we put point's coordinates in the previous expression:

$$\nabla\vec{\varphi}(1; -2; -1) = -12\vec{\imath} - 9\vec{\jmath} - 16\vec{k}$$

Answer: $\nabla\vec{\varphi}(1; -2; -1) = -12\vec{\imath} - 9\vec{\jmath} - 16\vec{k}$.

Question

3. Find unit normal to a surface: $x^2y + 2xz = 4$ at a point (2;-2;3)

Solution

First, we rewrite the surface equation in the form of F(x,y,z)=0.

$x^2y + 2xz - 4 = 0.$

A normal to a surface can be found as $\left(F'_x(A); F'_y(A); F'_z(A)\right)$, where A is the point (2;-2;3).

$$F'_x(A) = 2xy + 2z = -4$$
$$F'_y(A) = x^2 = 4$$
$$F'_z(A) = 2x = 4$$

Vector is (-4;4;4) or (23;-23;-23).

Answer: (-4;4;4) or (23;-23;-23).

Question

4. $\varphi = xy^2z$ $\vec{A} = xz\vec{\imath} - xy^2\vec{\jmath} + yz^2\vec{k}$. Find $\frac{\partial^3\varphi\vec{A}}{\partial x^2\partial z}$.

Solution

$\varphi\vec{A} = x^2y^2z^2\vec{\imath} - x^2y^4z\vec{\jmath} + xy^3z^3\vec{k}$

$$\frac{\partial^3(\varphi\vec{A})}{\partial x^2\partial z} = \frac{\partial^2\frac{\partial(\varphi\vec{A})}{\partial z}}{\partial x^2} = \frac{\partial^2(2x^2y^2z\vec{\imath} - x^2y^4\vec{\jmath} + 3xy^3z^2\vec{k})}{\partial x^2} = 4y^2z\vec{\imath} - 2y^4\vec{\jmath}$$

Answer should be based on point (not given in task), but most appropriate from answers given:

$\frac{\partial^3(\varphi\vec{A})}{\partial x^2\partial z} = 4y^2z\vec{\imath} - 2y^4\vec{\jmath} = 4\vec{\imath} - 2\vec{\jmath}$.

Question

5. $\varphi = 2x^2y - xz^3$. Find $\nabla^2\vec{\varphi}$.

Solution

$\nabla\vec{\varphi}=\frac{\partial\varphi}{\partial x}\vec{\imath} + \frac{\partial\varphi}{\partial y}\vec{\jmath} + \frac{\partial\varphi}{\partial z}\vec{k} = (4xy - z^3)\vec{\imath} + 2x^2\vec{\jmath} - 3xz^2\vec{k}$,

$$\nabla^2\vec{\varphi} = 16x^2y^2 + z^6 - 8xyz^3 + 4x^4 + 9x^2z^4$$

Answer: $\nabla^2\vec{\varphi} = 16x^2y^2 + z^6 - 8xyz^3 + 4x^4 + 9x^2z^4$

Question

6. $\vec{A} = xz^3\vec{\imath} - 2x^2yz\vec{\jmath} + 2yz\vec{k}$. Find$\left[\nabla * \vec{A}\right]$ at point (1;-1;1).

Solution

$\left[\nabla * \vec{A}\right] = \left(-\frac{\partial 2yz}{\partial z} - \frac{\partial 2x^2yz}{\partial y}\right)\vec{\imath} + \left(-\frac{\partial 2yz}{\partial x} + \frac{\partial xz^3}{\partial z}\right)\vec{\jmath} + \left(-\frac{\partial xz^3}{\partial y} - \frac{\partial 2x^2yz}{\partial x}\right)\vec{k} = (-2y - 2x^2z)\vec{\imath} + 3xz^2\vec{\jmath} - 4xyz\vec{k}$

Now we put point's coordinates $x = 1, y = -1, z = 1$ in the previous expression:

$\left[\nabla * \vec{A}\right] = 3\vec{\jmath} + 4\vec{k}$.

Answer: $\left[\nabla * \vec{A}\right] = 3\vec{\jmath} + 4\vec{k}$.

Question

7. $\vec{A} = A1\vec{\imath} + A2\vec{\jmath} + A3\vec{k}$ $\vec{r} = x\vec{\imath} + y\vec{\jmath} + z\vec{k}$. Evaluate $\vec{\nabla} \cdot (\vec{A} * \vec{r})$.

Solution

$\vec{\nabla} \cdot \left(\vec{A} * \vec{r}\right) = \overrightarrow{\nabla_A} \cdot \left(\vec{A} * \vec{r}\right) + \overrightarrow{\nabla_r} \cdot \left(\vec{A} * \vec{r}\right) = \vec{r} \cdot \left(\overrightarrow{\nabla_A} * \vec{A}\right) - \vec{A} \cdot \left(\overrightarrow{\nabla_r} * \vec{r}\right)$.

As $\vec{A}$ is constant $\overrightarrow{\nabla_A} * \vec{A}$=0. $\overrightarrow{\nabla_r} * \vec{r}$=0 by definition. So answer is 0.

Answer: $\vec{\nabla} \cdot \left(\vec{A} * \vec{r}\right) = 0$.

Question

8. $\vec{A} = x^2y\vec{\imath} - 2xz\vec{\jmath} + 2yz\vec{k}$ Find $Curl\ curl\ \vec{A}$.

Solution

$Curl\ curl\ \vec{A}\ is\ the\ \ same\ as$ $[\vec{\nabla} * [\vec{\nabla} * \vec{A}]]$.

$\left[\vec{\nabla} * \left[\vec{\nabla} * \vec{A}\right]\right] = grad\left(div\vec{A}\right) - \Delta\vec{A} = grad(2xy + 2y) - (2y\vec{\imath}) = 2y\vec{\imath} + (2x + 2)\vec{\jmath} - 2y\vec{\imath} =$
$= (2x + 2)\vec{\jmath}$.

Answer: $\left[\vec{\nabla} * \left[\vec{\nabla} * \vec{A}\right]\right] = (2x + 2)\vec{\jmath}$.

Question

9. $\vec{A} = 2x^2\vec{\imath} - 3yz\vec{\jmath} + xz^2\vec{k}$ $\varphi = 2z - x^3y$ Find $\vec{A} \cdot \nabla\varphi$ at point (1;-1;1).

Solution

$\nabla\varphi = -3x^2y\vec{\imath} - x^3\vec{\jmath} + 2\vec{k}$

$$\vec{A}\cdot\nabla\varphi = -6x^4y + 3yzx^3 + 2xz^2$$

Put point's coordinates $x = 1, y = -1, z = 1$ in the last expression:

$$\vec{A}\cdot\nabla\varphi(1;-1;1) = 6 - 3 + 2 = 5$$

Answer: $\vec{A}\cdot\nabla\varphi(1;-1;1) = 5.$

Question

10. Find the directional derivative of $\varphi = x^2yz + 4xz^2$ in direction $l = 2\vec{\imath} - \vec{\jmath} - 2\vec{k}$ at point (1;-2;-1)

Solution

$\nabla\varphi = (2xyz + 4z^2)\vec{\imath} + x^2z\vec{\jmath} + (x^2y + 8xz)\vec{k}$

Directional derivative is $l\cdot\nabla\varphi = 2(2xyz + 4z^2) - x^2z - 2(x^2y + 8xz)$

Putting values $x = 1, y = -2, z = -1$ in the previous expression

$l\cdot\nabla\varphi = 2(4 + 4) + 1 - 2(-2 - 8) = 16 + 1 + 20 = 37$

Answer: $l\cdot\nabla\varphi = 37.$