Answer on Question #57660 – Math – Functional Analysis

Q.1: Find the Fourier series of f(x)=|x| on [-π, π]

Q.2: Find the Fourier series of f(x)=2-x2 on (-2<x<2)

Solution

1.

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |\mathrm{x}|\,dx = \frac{1}{\pi}\int_{0}^{\pi} \mathrm{x}dx = \frac{\pi}{2}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} |\mathrm{x}|\cos nx\,dx = \frac{2}{\pi}\int_{0}^{\pi} \mathrm{x}\cos nx\,dx = \frac{2}{\pi}\left[\frac{\mathrm{x}\sin nx}{n} + \frac{\cos nx}{n^2}\right]_0^{\pi} = -2\frac{[1-(-1)^n]}{\pi n^2}.$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} |\mathrm{x}|\sin nx\,dx$$

Before we dive into the integration observe that $|\mathrm{x}|$ defines an even function while $\sin nx$ defines an odd function. Since the product of an even and an odd function is odd, it follows that $b_n = 0$ for all n. This simple observation saves us from having to perform a rather laborious integration.

The Fourier series is given by

$$f(x) \sim \frac{\pi}{2} - \sum_{n=1}^{\infty} 2\frac{[1-(-1)^n]}{\pi n^2}\cos nx.$$

2.

$$a_0 = \frac{1}{4}\int_{-2}^{2} 2dx - \frac{1}{4}\int_{-2}^{2} x^2 dx = 2\frac{1}{4}(x)_{-2}^{2} - \frac{1}{4}\left(\frac{1}{3}x^3\right)_{-2}^{2} = 2 - \frac{1}{12}\big(8-(-8)\big) = 2 - \frac{4}{3} = \frac{2}{3}.$$

$$a_n = \frac{1}{2}\int_{-2}^{2} 2\cos\left(\frac{\pi n x}{2}\right)dx - \frac{1}{2}\int_{-2}^{2} x^2\cos\left(\frac{\pi n x}{2}\right)dx = \frac{16}{\pi^2 n^2}(-1)^{n+1}.$$

$$\frac{1}{2}\int_{-2}^{2} 2\cos\left(\frac{\pi n x}{2}\right)dx = \int_{-2}^{2}\cos\left(\frac{\pi n x}{2}\right)dx = \frac{2}{n\pi}\left(\sin\left(\frac{\pi n x}{2}\right)\right)_{-2}^{2} = \frac{2}{n\pi}(\sin(\pi n) - \sin(-\pi n)) = \frac{2}{n\pi}(0-0)$$
$$= 0$$

$$\frac{1}{2}\int_{-2}^{2} x^2\cos\left(\frac{\pi n x}{2}\right)dx = \frac{1}{2}\left(\frac{2}{n\pi}\left(x^2\sin\left(\frac{\pi n x}{2}\right)\right)_{-2}^{2} - \frac{4}{n\pi}\int_{-2}^{2} x\sin\left(\frac{\pi n x}{2}\right)dx\right)$$

$$= \frac{1}{2}\left(\frac{8}{n\pi}(\sin(\pi n) - \sin(-\pi n)) - \frac{4}{n\pi}\left(-\frac{2}{n\pi}\left(x\cos\left(\frac{\pi n x}{2}\right)\right)_{-2}^{2} + \frac{2}{n\pi}\int_{-2}^{2}\cos\left(\frac{\pi n x}{2}\right)dx\right)\right)$$

$$= \frac{1}{2}\left(\frac{8}{n\pi}(0-0) - \frac{4}{n\pi}\left(-\frac{2}{n\pi}\left(x\cos\left(\frac{\pi n x}{2}\right)\right)_{-2}^{2} + \frac{2}{n\pi}\int_{-2}^{2}\cos\left(\frac{\pi n x}{2}\right)dx\right)\right).$$

We already know that

$$\int_{-2}^{2}\cos\left(\frac{\pi n x}{2}\right)dx = 0.$$

$$\frac{1}{2}\int_{-2}^{2} x^2\cos\left(\frac{\pi n x}{2}\right)dx = \frac{4}{\pi^2 n^2}(2\cos\pi n - (-2)\cos(-\pi n)) = \frac{16}{\pi^2 n^2}(-1)^n.$$

Thus

$$a_n = 0 - \frac{16}{\pi^2 n^2}(-1)^n = \frac{16}{\pi^2 n^2}(-1)^{n+1}.$$

$$b_n = -\frac{1}{2}\int_{-2}^{2}(2 - x^2)\sin\left(\frac{\pi n x}{2}\right)dx$$

Before we dive into the integration observe that $2 - x^2$ defines an even function while $\sin\left(\frac{\pi n x}{2}\right)$ defines an odd function. Since the product of an even and an odd function is odd, it follows that $b_n = 0$ for all n. This simple observation saves us from having to perform a rather laborious integration.

The Fourier series is given by

$$f(x) \sim \frac{2}{3} + \sum_{n=1}^{\infty}\frac{16}{\pi^2 n^2}(-1)^{n+1}\cos\left(\frac{\pi n x}{2}\right).$$