Answer on Question #56773 – Math – Combinatorics | Number Theory

How many words can we build using exactly 5 A's, 5 B's and 5 C's if the first 5 letters cannot be A's, the second 5 letters cannot be B's and the third 5 letters cannot be C's? Hint: Group the different ways according to the number of B's in the first group.

Solution

Assume that exactly N A's stand on the last 5 positions (so $N \in \{0,1,2,3,4,5\}$). It follows that there are $5 - N$ B's on the last 5 positions and $5 - N$ A's on the middle 5 positions. Also it follows that there are N C's on the middle 5 positions and $5 - N$ C's on the first 5 positions.

So there are $\binom{5}{N}$ ways to place N A's on the last 5 positions. Then there are $\binom{5}{5-N}$ ways to place the rest A's on the middle 5 positions. Finally, there are $\binom{5}{N}$ ways to place N B's on the first 5 positions. So if we place N A's on the last 5 positions, $5 - N$ A's on the middle 5 positions and N B's on the first 5 positions, then the rest positions can be filled in the only way – C's on the first and on the middle 5 positions, B's on the last 5 positions.

Hence the number of words with N A's on the last 5 positions is equal to

$$\binom{5}{N} \cdot \binom{5}{5-N} \cdot \binom{5}{N} = \binom{5}{N}^3$$

and the total number is

$$\sum_{N=0}^{5} \binom{5}{N}^3 = 1^3 + 5^3 + 10^3 + 10^3 + 5^3 + 1^3 = 2252.$$

Answer: 2252.