**Answer on Question #56762 – Math – Combinatorics | Number Theory**

5 balls are to be placed in 3 boxes. Each box can hold all 5 balls. In how many different ways can we place the balls so that no box remains empty if,

**(i)** balls & boxes are different
**(ii)** balls are different but boxes are identical
**(iii)** balls are identical but boxes are different
**(iv)** balls as well as boxes are identical

## Solution

**(i) balls and boxes are different.**

Because

5=2+2+1, 5=3+1+1,

we can have the following two distributions:
**(a)** 3-1-1: one box gets three balls and the remaining two boxes get one ball each.
The number of ways to distribute the balls for this case is

$$3 * 2 * C_5^3 = 60,$$

where 3 is the number of ways to choose which box gets 3 balls (we have 3 boxes, thus 3 choices for that), $C_5^3 = \binom{5}{3} = \frac{5!}{3!2!}$ is the number of ways to choose which 3 balls out of 5 will go to that box, and 2 is the number of ways to distribute the remaining 2 balls in the remaining two boxes.
**(b)** 2-2-1: one box gets one ball and the remaining two boxes get two balls each.
The number of ways to distribute the balls for this case is

$$3 * 5 * C_4^2 = 90,$$

where 3 is the number of ways to choose which box gets 1 balls (we have 3 boxes, thus 3 choices for that), 5 is the number of ways to choose which ball out of 5 will go to that box, and $C_4^2 = \frac{4!}{2!2!}$ is the number of ways to choose which 2 balls out of

4 balls left will go to the second box (the remaining 2 balls will naturally go to the third box).

Thus, the total number of required ways is

$$N = 60 + 90 = 150.$$

**(ii) balls are different but boxes are identical.**

When the boxes are identical, the distributions of 1, 1, 3 balls, 1, 3, 1 balls and 3, 1, 1 balls will be treated as identical distributions. Number of ways of distributing in this manner is

$$\frac{5!}{1!1!3!} = 20.$$

When the boxes are identical, the distributions of 1, 2, 2 balls, 2, 1, 2 balls and 2, 2, 1 balls will be treated as identical distributions. Number of ways of distributing in this manner is

$$\frac{5!}{2!2!1!} = 30.$$

Thus, the total number of required ways is

$$N = 20 + 30 = 50.$$

**(iii) balls are identical but boxes are different .**

First, consider the case, where empty box is allowed.

Total number of ways of distributing $r$ identical balls in $n$ different boxes is the same as the number of $r$-combinations of $n$ items, repetitions are allowed.

It is

$$C(n + r - 1, r) = \binom{n + r - 1}{r} = \binom{n + r - 1}{n - 1}.$$

Now consider the case, where no box is empty.

First take $n = 3$ balls and put one ball in each box. This leaves

$r - n = 5 - 3 = 2$ balls to distribute with no restrictions as in the previous case.

Thus, there are

$$\binom{(r-n)+n-1}{n-1} = \binom{r-1}{n-1} = \binom{5-1}{3-1} = \binom{4}{2} = \frac{4!}{2!2!} = \frac{4\cdot 3}{2} = \frac{12}{2} = 6$$

ways.

**(iv) balls as well as boxes are identical.**

It is the number of partitions of $r$ into $n$ parts, that is, write $r$ as a sum of natural numbers, order is unimportant.

For $r = 5, n = 3$ the partitions are

5=2+2+1,

5=3+1+1.

The partition 2+2+1 says put 2 balls in the first box, 2 balls in the second box and one ball in the third box. The partition 3+1+1 says put 3 balls in the first box, 1 ball in the second box and 1 ball in the third box.

We have only 2 partitions, hence the total number of ways is 2.

**Answer: (i) 150; (ii) 50; (iii) 6; (iv) 2.**