Answer on Question #55335 – Math – Statistics and Probability

Question

10 red marbles and 10 blue marbles are placed into a bag. Alex mixes up the bag and randomly selects a marble. He continues to do so, replacing the marble after each selection, until a red marble is selected.
a. What is the probability that the first time that a red marble is pulled is on Alex's 6th try?
b. On average, how many marbles will Alex have to pull in order to get a red marble? (Hint: use math expectation).

Solution

a. Plainly, there are 10+10=20 marbles in the bag. Using the classical definition of probability, the probability of selecting a red marble is given by

$$p = \frac{10}{20} = 0.5.$$

Similarly, probability of selecting a blue marble is

$$q = \frac{10}{20} = 0.5 = 1 - p.$$

If marbles are replaced, then probability remains the same for all these experiments. We are asked what are the odds of selecting a red marble for the first time on the 6th try. It means that we are asked to determine a probability of the case we will call Q (Alex selects 5 blue marbles on five first tries and a red one on the 6-th try).
Events of marble selection are independent, therefore

$$P(Q) = P(1^{st} = blue\ and\ 2^{nd} = blue\ and\ \ldots and\ 5^{th} = blue\ and\ 6^{th} = red) =$$
$$= P(1^{st} = blue\ ) \cdot P(2^{nd} = blue) \cdot \ldots \cdot P(5^{th} = blue) \cdot P(6^{th} = red) =$$
$$= q \cdot q \cdot \ldots \cdot q \cdot\ p =\ q^5 \cdot p = 0.5^5 \cdot 0.5 \approx 0.016 \text{ or } P(Q) = 1.6\%$$

Answer: 0.016.

Solution

b. Average number of pulls (denoted by E(X)) is a mathematical expectation of number of pulls. If Alex pulls a red marble on the 1st try, then the number of such pulls will be

$$x_1 = 1.$$

Probability of this event is just

$$p_1 = p = 0.5.$$

Now, if it happens on the 2nd try, the number is

$$x_2 = 2,$$

and the probability is

$$p_2 = q \cdot p = p^2,$$

because this event is the following:
Alex selecting a blue marble on the first try (with probability q) and then selecting a red one (with probability p). Now, for 3-rd try the number of pulls is

$$x_3 = 3,$$

and the probability is

$$p_3 = q \cdot q \cdot p = p^3.$$

Observing the pattern, we get formulae for the n-th try:

number of pulls is

$$x_n = n$$

and probability is

$$p_n = q \cdot \ldots \cdot q \cdot p = q^{n-1} \cdot p = p^n.$$

Then, by definition of the math expectation, we have

$$\mathrm{E(X)} = \sum_{k=1}^{\infty} x_k p_k = x_1 p_1 + x_2 p_2 + x_3 p_3 + \cdots = \sum_{k=1}^{\infty} k\, p^k = \frac{p}{(1-p)^2} = \frac{0.5}{(1-0.5)^2} = \frac{1}{0.5} = 2.$$

Answer: 2.