Answer on Question #54091-Math-Integral Calculus

State and prove fundamental theorem of integral calculus.

Solution

The first fundamental theorem says that the integral of the derivative is the function; or, more precisely, that it's the difference between two outputs of that function.

Theorem: (First Fundamental Theorem of Calculus) If f is continuous and $F' = f$, then

$$\int_a^b f(x)dx = F(b) - F(a).$$

Proof: By using Riemann sums, we will define an antiderivative G of f and then use G(x) to calculate $F(b) - F(a)$. We start with the fact that $F' = f$ and f is continuous. (It's not strictly necessary for f to be continuous, but without this assumption we can't use the second fundamental theorem in our proof.)

Next, we define $G(x) = \int_a^x f(t)\,dt$. (We know that this function exists because we can define it using Riemann sums.)

The second fundamental theorem of calculus tells us that:

$$G'(x) = f(x)$$

So $F'(x) = G'(x)$. Therefore,

$$(F - G)' = F' - G' = f - f = 0$$

From the mean value theorem if two functions have the same derivative then they differ only by a constant, so F − G = constant or

$$F(x) = G(x) + c.$$

This is an essential step in an essential proof; all of calculus is founded on the fact that if two functions have the same derivative, they differ by a constant.

Now we compute $F(b) - F(a)$ to see that it is equal to the definite integral:

$$F(b) - F(a) = (G(b) + c) - (G(a) + c) = G(b) - G(a) = \int_a^b f(t)dt - \int_a^a f(t)dt = \int_a^b f(t)dt - 0$$

$$F(b) - F(a) = \int_a^b f(x)dx.$$