Answer on Question #53985 – Math – Calculus

Question

If $t = \tan\left(\frac{x}{2}\right)$,

then

$$\cos x = \frac{1-t^2}{1+t^2}, \qquad \sin x = \frac{2t}{1+t^2}.$$

Solution

We have

$$\frac{1-t^2}{1+t^2} = \frac{1-\left(\tan\left(\frac{x}{2}\right)\right)^2}{1+\left(\tan\left(\frac{x}{2}\right)\right)^2} = \frac{1-\left(\frac{\sin\left(\frac{x}{2}\right)}{\cos\left(\frac{x}{2}\right)}\right)^2}{1+\left(\frac{\sin\left(\frac{x}{2}\right)}{\cos\left(\frac{x}{2}\right)}\right)^2} = \frac{\frac{\left(\cos\left(\frac{x}{2}\right)\right)^2-\left(\sin\left(\frac{x}{2}\right)\right)^2}{\left(\cos\left(\frac{x}{2}\right)\right)^2}}{\frac{\left(\cos\left(\frac{x}{2}\right)\right)^2+\left(\sin\left(\frac{x}{2}\right)\right)^2}{\left(\cos\left(\frac{x}{2}\right)\right)^2}} =$$

$$= \frac{\cos\left(2\cdot\frac{x}{2}\right)}{\left(\cos\left(\frac{x}{2}\right)\right)^2}\cdot\frac{\left(\cos\left(\frac{x}{2}\right)\right)^2}{\left(\cos\left(\frac{x}{2}\right)\right)^2+\left(\sin\left(\frac{x}{2}\right)\right)^2} = \frac{\cos x}{1} = \cos x\,;$$

$$\frac{2t}{1+t^2} = \frac{2\tan\left(\frac{x}{2}\right)}{1+\left(\tan\left(\frac{x}{2}\right)\right)^2} = \frac{\frac{2\sin\left(\frac{x}{2}\right)}{\cos\left(\frac{x}{2}\right)}}{1+\left(\frac{\sin\left(\frac{x}{2}\right)}{\cos\left(\frac{x}{2}\right)}\right)^2} = \frac{2\sin\left(\frac{x}{2}\right)}{\cos\left(\frac{x}{2}\right)}\cdot\frac{\left(\cos\left(\frac{x}{2}\right)\right)^2}{\left(\cos\left(\frac{x}{2}\right)\right)^2+\left(\sin\left(\frac{x}{2}\right)\right)^2} =$$

$$= \frac{2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)}{1} = \sin\left(2\cdot\frac{x}{2}\right) = \sin x.$$

So

$$\cos x = \frac{1-t^2}{1+t^2},\ \sin x = \frac{2t}{1+t^2},$$

where $t = \tan\left(\frac{x}{2}\right)$.