Answer on Question #53419 – Math – Integral Calculus

Question

Find the area of the region that lies inside the first curve and outside the second curve. r2 = 18 cos 2θ, r = 3

Solution

Definition

Let G be the region bounded by curves with polar equations $r=f(\vartheta)$ and $r=g(\vartheta)$ (fig.1), $\vartheta=a$ and $\vartheta=b$, where $f(\vartheta)\geq g(\vartheta)>0$, $0<b-a\leq 2\pi$. Then area A of G is

$$A(G) = \frac{1}{2}\int_a^b ([f(\vartheta)]^2 - [g(\vartheta)]^2)d\vartheta. \quad (1)$$

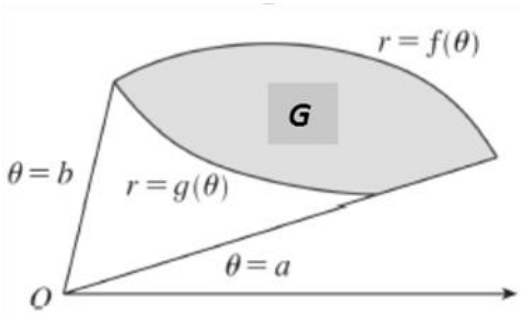

Fig.1

First we describe the given curves in polar coordinates $\{r,\vartheta\}$:

1) $r^2 = 18\cos(2\vartheta)$ is the lemniscate of Bernoulli;
2) $r = 3$ is a circle with radius 3 and the center at the pole (origin).

Further we sketch these curves (fig.2). The required area is represented by the shaded regions.

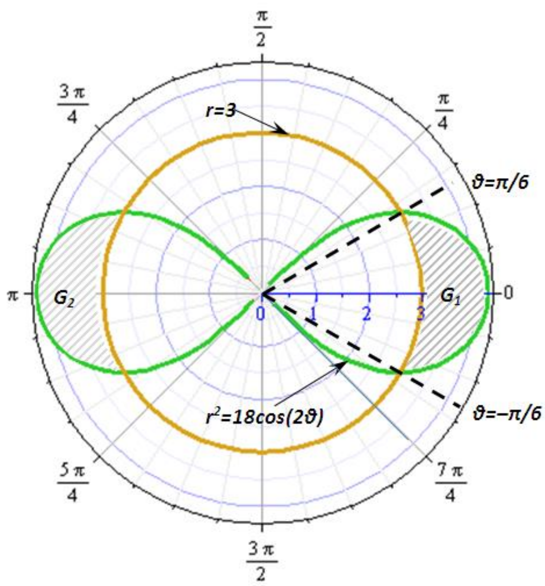

Fig.2

As we can see,

$$A(G) = A(G_1) + A(G_2). \qquad (2)$$

As the figures in the fig.2 are symmetrical, then we can write

$$A(G) = 2A(G_1). \qquad (3)$$

Let`s find points of intersection of the curves 1) and 2) for region G_1:

$$\pm\sqrt{18\cos(2\vartheta)} = 3,$$

$$18\cos(2\vartheta) = 9,$$

$$\cos(2\vartheta) = \frac{9}{18} = \frac{1}{2},$$

$$2\vartheta = \pm\arccos\left(\frac{1}{2}\right) = \pm\frac{\pi}{3},$$

$$\vartheta_{1,2} = \pm\frac{\pi}{6}.$$

Therefore, using (1), (2) and (3) we obtain

$$A(G) = 2\cdot\frac{1}{2}\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}}(18\cos(2\vartheta) - 3^2)d\vartheta = \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}}(18\cos(2\vartheta) - 9)d\vartheta = \left(\frac{18}{2}\sin(2\vartheta) - 9\vartheta\right)\Big|_{-\frac{\pi}{6}}^{\frac{\pi}{6}}$$

$$= 9\left(\sin\left(2\cdot\frac{\pi}{6}\right) - \left(\sin\left(2\cdot\left(-\frac{\pi}{6}\right)\right)\right) - \left(\frac{\pi}{6} - \left(-\frac{\pi}{6}\right)\right)\right) = 9\left(2\sin\left(2\cdot\frac{\pi}{6}\right) - \frac{2\pi}{6}\right)$$

$$= 18\left(\sin\left(\frac{\pi}{3}\right) - \frac{\pi}{6}\right) = 18\left(\frac{\sqrt{3}}{2} - \frac{\pi}{6}\right) \approx 6.16\ square\ units.$$

Answer: *A(G)=6.16 square units.*