**Answer on Question #53070 – Math – Integral Calculus**

Integrate:

$$\int \frac{dx}{a + bsinx}$$

**Solution**

Let

$$t = tan\frac{x}{2}$$

By the double-angle formula for the sine function

$$sin\, x = 2\sin\frac{x}{2}\, cos\frac{x}{2} = 2tcos^2\frac{x}{2} = \frac{2t}{sec^2\frac{x}{2}} = \frac{2t}{1 + t^2}$$

The differential $dx$ can be calculated as follows:

$$\frac{dt}{dx} = \frac{1}{2}sec^2\frac{x}{2} = \frac{1 + t^2}{2}$$

$$dx = \frac{2}{1 + t^2}dt$$

Substitute

$$\int \frac{dx}{a + bsinx} = \int \frac{\frac{2}{1+t^2}dt}{a + b\frac{2t}{1+t^2}} = \int \frac{\frac{2}{1+t^2}dt}{\frac{a + at^2 + 2bt}{1+t^2}} = \int \frac{2dt}{a + at^2 + 2bt}$$

$$= \int \frac{2dt}{a\left(t^2 + \frac{2b}{a}t\right) + a} = \int \frac{2dt}{a\left(t^2 + 2\frac{b}{a}t + \left(\frac{b}{a}\right)^2 - \left(\frac{b}{a}\right)^2\right) + a}$$

$$= \int \frac{2dt}{a\left(\left(t + \frac{b}{a}\right)^2 - \left(\frac{b}{a}\right)^2\right) + a}$$

$$= \int \frac{2dt}{a\left(t + \frac{b}{a}\right)^2 - \frac{b^2}{a} + a} = \frac{2}{a}\int \frac{dt}{\left(t + \frac{b}{a}\right)^2 + 1 - \left(\frac{b}{a}\right)^2}$$

$$= \frac{2}{a}\int \frac{d\left(t + \frac{b}{a}\right)}{\left(t + \frac{b}{a}\right)^2 + 1 - \left(\frac{b}{a}\right)^2} = \frac{2}{a}\int \frac{d\left(t + \frac{b}{a}\right)}{\left(t + \frac{b}{a}\right)^2 + 1 - \left(\frac{b}{a}\right)^2}$$

Now we have 3 different cases:

1) If $1 - \left(\frac{b}{a}\right)^2 \geq 0$, which is equivalent to $a^2 \geq b^2$

$$I=\frac{2}{a}\int\frac{d\left(t+\frac{b}{a}\right)}{\left(t+\frac{b}{a}\right)^2+1-\left(\frac{b}{a}\right)^2}=\frac{2}{a}\frac{1}{\sqrt{1-\left(\frac{b}{a}\right)^2}}arctan\left(\frac{t+\frac{b}{a}}{\sqrt{1-\left(\frac{b}{a}\right)^2}}\right)+C=\frac{2}{\sqrt{a^2-b^2}}arctan\left(\frac{a\,tan\frac{x}{2}+b}{\sqrt{a^2-b^2}}\right)+$$

$+C,$

where $C$ is an arbitrary real constant.

2) If $1-\left(\frac{b}{a}\right)^2=0 \quad => \quad a^2=b^2$

$$I=\frac{2}{a}\int\frac{d\left(t+\frac{b}{a}\right)}{\left(t+\frac{b}{a}\right)^2+1-\left(\frac{b}{a}\right)^2}=\frac{2}{a}\int\frac{d\left(t+\frac{b}{a}\right)}{\left(t+\frac{b}{a}\right)^2}=\frac{2}{a}\int\left(t+\frac{b}{a}\right)^{-2}d\left(t+\frac{b}{a}\right)=\frac{2}{a}\frac{\left(t+\frac{b}{a}\right)^{-2+1}}{(-2+1)}$$

$$+C=\frac{-2}{a\left(t+\frac{b}{a}\right)}+C=\frac{-2}{a\,tan\frac{x}{2}+b}+C$$

3) If $1-\left(\frac{b}{a}\right)^2\le 0 \quad => a^2\le b^2$

$$I=\frac{2}{a}\int\frac{d\left(t+\frac{b}{a}\right)}{\left(t+\frac{b}{a}\right)^2+1-\left(\frac{b}{a}\right)^2}=\frac{2}{a}\frac{1}{2}ln\left|\frac{t+\frac{b}{a}-\left(\left(\frac{b}{a}\right)^2-1\right)}{t+\frac{b}{a}+\left(\left(\frac{b}{a}\right)^2-1\right)}\right|+C$$

$$=\frac{1}{a}ln\left|\frac{tan\frac{x}{2}+\frac{b}{a}-\left(\left(\frac{b}{a}\right)^2-1\right)}{tan\frac{x}{2}+\frac{b}{a}+\left(\left(\frac{b}{a}\right)^2-1\right)}\right|+C$$

**Answer:** $\frac{2}{\sqrt{a^2-b^2}}arctan\left(\frac{a\,tan\frac{x}{2}+b}{\sqrt{a^2-b^2}}\right)+c$, if $a^2>b^2$

$\frac{-2}{a\,tan\frac{x}{2}+b}+C,$ if $a^2=b^2$

$\frac{1}{a}ln\left|\frac{tan\frac{x}{2}+\frac{b}{a}-\left(\left(\frac{b}{a}\right)^2-1\right)}{tan\frac{x}{2}+\frac{b}{a}+\left(\left(\frac{b}{a}\right)^2-1\right)}\right|+C,$ if $a^2<b^2$