Answer on Question #53068 – Math – Integral Calculus

Integrate $\int \frac{dx}{a+b\cos x}$

Solution

$$I=\int \frac{dx}{a+b\cos x}=\left| \begin{matrix} t=\tan \frac{x}{2} \\ dx=\frac{2dt}{1+t^2} \\ \cos x=\frac{1-t^2}{1+t^2} \end{matrix} \right| =\int \frac{\frac{2dt}{1+t^2}}{a+b\frac{1-t^2}{1+t^2}}=2\int \frac{dt}{a(1+t^2)+b(1-t^2)}=2\int \frac{dt}{(a+b)+(a-b)t^2}$$

This integral depends on the parameters a and b. It's obvious that

$$I=I(a,b)=\begin{cases} I_1=2\int \frac{dt}{2a}, if\ b=a \\ I_2=2\int \frac{dt}{2at^2}, if\ b=-a \\ I_3=2\int \frac{dt}{(a+b)+(a-b)t^2}, if\ b\neq a\ and\ b\neq -a \end{cases}$$

$$I_1=\int \frac{dx}{a+a\cos x}=\left| \begin{matrix} t=\tan \frac{x}{2} \\ dx=\frac{2dt}{1+t^2} \\ \cos x=\frac{1-t^2}{1+t^2} \end{matrix} \right| =2\int \frac{dt}{a(1+t^2)+a(1-t^2)}=2\int \frac{dt}{2a}=\frac{2}{2a}\int dt=\frac{1}{a}t+c=\frac{1}{a}\tan \frac{x}{2}+c$$

$$I_2=\int \frac{dx}{a-a\cos x}=\left| \begin{matrix} t=\tan \frac{x}{2} \\ dx=\frac{2dt}{1+t^2} \\ \cos x=\frac{1-t^2}{1+t^2} \end{matrix} \right| =2\int \frac{dt}{a(1+t^2)-a(1-t^2)}=2\int \frac{dt}{2at^2}=\frac{2}{2a}\int \frac{1}{t^2}dt=-\frac{1}{a}\frac{1}{t}+c=-\frac{1}{a}\frac{1}{\tan \frac{x}{2}}+c$$

$$I_3=\int \frac{dx}{a+b\cos x}=\left| \begin{matrix} t=\tan \frac{x}{2} \\ dx=\frac{2dt}{1+t^2} \\ \cos x=\frac{1-t^2}{1+t^2} \end{matrix} \right| =2\int \frac{dt}{a(1+t^2)+b(1-t^2)}=2\int \frac{dt}{(a+b)+(a-b)t^2}=\frac{2}{a-b}\int \frac{dt}{\frac{a+b}{a-b}+t^2}=$$

$$=\begin{cases} I_{31}, if\ \mathrm{a}>b \\ I_{32}, if\ a<b \end{cases}$$

$$I_{32}=\frac{2}{a-b}\int \frac{dt}{t^2-\gamma^2}=\left| \gamma^2=-\frac{(a+b)}{(a-b)}=\frac{(a+b)}{(b-a)}>0 \right|$$ To evaluate I_{32} apply the Partial Fraction Decomposition Method.

$$\frac{1}{t^2-\gamma^2}=\frac{A}{t-\gamma}+\frac{B}{t+\gamma}=\frac{A(t+\gamma)+B(t-\gamma)}{t^2-\gamma^2}=\frac{(A+B)t+(A-B)\gamma}{t^2-\gamma^2}\Rightarrow$$

$$\begin{cases} A+B=0 \\ A-B=\frac{1}{\gamma} \end{cases}\Rightarrow\begin{cases} A=\frac{1}{2\gamma} \\ B=-\frac{1}{2\gamma} \end{cases}$$

$$\frac{1}{t^2-\gamma^2}=\frac{1}{2\gamma}\frac{1}{t-\gamma}-\frac{1}{2\gamma}\frac{1}{t+\gamma}$$

$$I_{32}=\frac{2}{a-b}\int\frac{dt}{t^2-\gamma^2}=\frac{2}{a-b}\frac{1}{2\gamma}\int\frac{dt}{t-\gamma}-\frac{2}{a-b}\frac{1}{2\gamma}\int\frac{dt}{t+\gamma}=\frac{1}{a-b}\frac{1}{\gamma}\ln|t-\gamma|-\frac{1}{a-b}\frac{1}{\gamma}\ln|t+\gamma|+c=$$

$$=\frac{1}{a-b}\frac{1}{\gamma}\ln\left|\frac{t-\gamma}{t+\gamma}\right|+c=\frac{1}{a-b}\frac{1}{\sqrt{\frac{(a+b)}{(b-a)}}}\ln\left|\frac{\tan\frac{x}{2}-\sqrt{\frac{(a+b)}{(b-a)}}}{\tan\frac{x}{2}+\sqrt{\frac{(a+b)}{(b-a)}}}\right|+c=$$

$$=-\frac{1}{b-a}\sqrt{\frac{b-a}{a+b}}\ln\left|\frac{\tan\frac{x}{2}-\sqrt{\frac{(a+b)}{(b-a)}}}{\tan\frac{x}{2}+\sqrt{\frac{(a+b)}{(b-a)}}}\right|+c=-\frac{1}{\sqrt{b^2-a^2}}\ln\left|\frac{\tan\frac{x}{2}-\sqrt{\frac{(a+b)}{(b-a)}}}{\tan\frac{x}{2}+\sqrt{\frac{(a+b)}{(b-a)}}}\right|+c$$

c is the integration constant.

$$I_{31}=\frac{2}{a-b}\int\frac{dt}{\left(\sqrt{\frac{(a+b)}{(a-b)}}\right)^2+t^2}=\frac{2}{a-b}\sqrt{\frac{(a-b)}{(a+b)}}\operatorname{arctan}\left(t\sqrt{\frac{(a-b)}{(a+b)}}\right)+c=$$

$$=\frac{2}{a-b}\sqrt{\frac{(a-b)}{(a+b)}}\operatorname{arctan}\left(\sqrt{\frac{(a-b)}{(a+b)}}\tan\frac{x}{2}\right)+c=$$

$$=\frac{2}{\sqrt{(a-b)(a+b)}}\operatorname{arctan}\left(\sqrt{\frac{(a-b)}{(a+b)}}\tan\frac{x}{2}\right)+c=\frac{2}{\sqrt{(a^2-b^2)}}\operatorname{arctan}\left(\sqrt{\frac{(a-b)}{(a+b)}}\tan\frac{x}{2}\right)+c$$

Answer: $I=I(a,b)=\begin{cases} \frac{1}{a}\tan\frac{x}{2}+c\ if\ b=a \\ -\frac{1}{a}\frac{1}{\tan\frac{x}{2}}+c\ if\ b=-a \\ \frac{2}{\sqrt{(a^2-b^2)}}\operatorname{arctan}\left(\sqrt{\frac{(a-b)}{(a+b)}}\cdot\tan\frac{x}{2}\right)+c,if\ \ a>b \\ -\frac{1}{\sqrt{b^2-a^2}}\ln\left|\frac{\tan\frac{x}{2}-\sqrt{\frac{(a+b)}{(b-a)}}}{\tan\frac{x}{2}+\sqrt{\frac{(a+b)}{(b-a)}}}\right|+c,if\ \ a<b \end{cases}$