Answer on Question #52820 – Math – Trigonometry

Verify that the following equality is an identity:

$$\sin\alpha\cdot\sin(60^\circ-\alpha)\cdot\sin(60^\circ+\alpha)=\frac{1}{4}\sin(3\alpha) \qquad (1)$$

Solution:

Let's consider the left side of the equality $z_l=\sin\alpha\cdot\sin(60^\circ-\alpha)\cdot\sin(60^\circ+\alpha)$.

First let's simplify z_l using the sum and difference formulas.

$$\sin(60^\circ-\alpha)=\sin(60^\circ)\cdot\cos\alpha-\cos(60^\circ)\cdot\sin\alpha=\frac{\sqrt{3}}{2}\cos\alpha-\frac{1}{2}\sin\alpha$$

$$\sin(60^\circ+\alpha)=\sin(60^\circ)\cdot\cos\alpha+\cos(60^\circ)\cdot\sin\alpha=\frac{\sqrt{3}}{2}\cos\alpha+\frac{1}{2}\sin\alpha$$

$$z_l=\sin\alpha\cdot\sin(60^\circ-\alpha)\cdot\sin(60^\circ+\alpha)=$$
$$=\sin\alpha\cdot(\frac{\sqrt{3}}{2}\cos\alpha-\frac{1}{2}\sin\alpha)\cdot(\frac{\sqrt{3}}{2}\cos\alpha-\frac{1}{2}\sin\alpha)=$$
$$=\sin\alpha\cdot(\frac{3}{4}\cos^2\alpha-\frac{1}{4}\sin^2\alpha)=\frac{1}{4}\sin\alpha\cdot(3\cos^2\alpha-\sin^2\alpha)=$$
$$=\frac{1}{4}\sin\alpha\cdot(3\cdot\underbrace{\cos^2\alpha}_{1-\sin^2\alpha}-\sin^2\alpha)=\frac{1}{4}\sin\alpha\cdot(3\cdot(1-\sin^2\alpha)-\sin^2\alpha)=$$
$$=\frac{1}{4}\sin\alpha\cdot(3-3\sin^2\alpha-\sin^2\alpha)=\frac{1}{4}(3\sin\alpha-4\sin^3\alpha)$$

Next, let's consider the right side of the equality $z_r=\frac{1}{4}\sin 3\alpha$.

$$z_r=\frac{1}{4}\sin(3\alpha)=\frac{1}{4}\sin(\alpha+2\alpha)=$$
$$=\frac{1}{4}(\sin\alpha\cos 2\alpha+\cos\alpha\sin 2\alpha)=\left|\begin{matrix}\cos 2\alpha=\cos^2\alpha-\sin^2\alpha=1-2\sin^2\alpha\\ \sin 2\alpha=2\sin\alpha\cos\alpha\end{matrix}\right|=$$
$$=\frac{1}{4}(\sin\alpha\cdot(1-2\sin^2\alpha)+2\cos^2\alpha\sin\alpha)=\frac{1}{4}(\sin\alpha\cdot(1-2\sin^2\alpha)+2(1-\sin^2\alpha)\sin\alpha)=$$
$$=\frac{1}{4}(\sin\alpha-2\sin^3\alpha+2\sin\alpha\cdot(1-\sin^2\alpha))=\frac{1}{4}(\sin\alpha-2\sin^3\alpha+2\sin\alpha-2\sin^3\alpha)=$$
$$=\frac{1}{4}(3\sin\alpha-4\sin^3\alpha)$$

The given equality (1) is an identity, because the left side z_l is equal to the right side z_r for any real value of α.