Answer on Question #51705 – Math - Integral Calculus

What is the integration of x^4/[{x^2+a^2}{x^2+b^2}]?

Solution.

$$I=\int\frac{x^4dx}{(x^2+a^2)(x^2+b^2)}$$

Step 1. Let's consider the integrand, i.e. the function that is to be integrated, $f(x)$:

$$f(x)=\frac{x^4}{(x^2+a^2)(x^2+b^2)}=\frac{x^4}{x^4+x^2(a^2+b^2)+a^2b^2}=$$

$$=\frac{x^4+(x^2(a^2+b^2)+a^2b^2)-(x^2(a^2+b^2)+a^2b^2)}{x^4+x^2(a^2+b^2)+a^2b^2}=$$

$$=1-\frac{x^2(a^2+b^2)+a^2b^2}{x^4+x^2(a^2+b^2)+a^2b^2}=1-\frac{x^2(a^2+b^2)+a^2b^2}{(x^2+a^2)(x^2+b^2)}$$

Step 2. Let's use the method of Partial Fraction Decomposition.

$$\frac{x^2(a^2+b^2)+a^2b^2}{(x^2+a^2)(x^2+b^2)}=\frac{Ax+B}{(x^2+a^2)}+\frac{Cx+D}{(x^2+b^2)}=$$

$$=\frac{Ax(x^2+b^2)+B(x^2+b^2)+Cx(x^2+a^2)+D(x^2+a^2)}{(x^2+a^2)(x^2+b^2)}=$$

$$=\frac{Ax^3+Bx^2+Axb^2+Bb^2+Cx^3+Dx^2+Cxa^2+Da^2}{(x^2+a^2)(x^2+b^2)}=$$

$$=\frac{x^3(A+C)+x^2(B+D)+x(Ab^2+Ca^2)+Bb^2+Da^2}{(x^2+a^2)(x^2+b^2)}$$

Step 3. Equate the left-hand and the right-hand sides, multiply through by the denominator so we no longer deal with fractions.

$$x^2(a^2+b^2)+a^2b^2=x^3(A+C)+x^2(B+D)+x(Ab^2+Ca^2)+Bb^2+Da^2$$

Step 4. Find unknown coefficients A, B, C and D, equating like powers of x in the last line. Now we have the system of four equations and solving them gives us unknown coefficients A, B, C and D.

$$
\begin{array}{ll}
x^3: & A+C=0 \\
x^2: & B+D=a^2+b^2 \\
x^1: & Ab^2+Ca^2=0 \\
x^0: & Bb^2+Da^2=a^2b^2
\end{array}
\Rightarrow
\begin{array}{c}
A=-C \\
Bb^2+Db^2=a^2b^2+b^4 \\
A=-\dfrac{Ca^2}{b^2} \\
Bb^2+Da^2=a^2b^2
\end{array}
\Rightarrow
\begin{array}{c}
A=0 \\
C=0 \\
Bb^2+Db^2=a^2b^2+b^4 \\
Bb^2+Da^2=a^2b^2
\end{array}
\Rightarrow
$$

$$
\Rightarrow
\begin{array}{l}
A=0 \\
C=0 \\
D=\dfrac{b^4}{b^2-a^2} \\
B=-\dfrac{a^4}{b^2-a^2}
\end{array}
$$

Step 5. Rewrite the integrand substituting the unknown coefficients A, B, C and D:

$$
f(x)=1-\frac{x^2(a^2+b^2)+a^2b^2}{(x^2+a^2)(x^2+b^2)}=1-\frac{Ax+B}{(x^2+a^2)}-\frac{Cx+D}{(x^2+b^2)}=
$$

$$
=1-\frac{B}{(x^2+a^2)}-\frac{D}{(x^2+b^2)}=1+\frac{a^4}{(b^2-a^2)}\frac{1}{(x^2+a^2)}-\frac{b^4}{(b^2-a^2)}\frac{1}{(x^2+b^2)}.
$$

Step 6. Compute the initial indefinite Integral:

$$
I=\int\frac{x^4dx}{(x^2+a^2)(x^2+b^2)}=\int dx\left(1+\frac{a^4}{(b^2-a^2)}\frac{1}{(x^2+a^2)}-\frac{b^4}{(b^2-a^2)}\frac{1}{(x^2+b^2)}\right)=
$$

$$
=x+\frac{a^4}{(b^2-a^2)}\frac{1}{a}\arctan\frac{x}{a}-\frac{b^4}{(b^2-a^2)}\frac{1}{b}\arctan\frac{x}{b}+c=
$$

$$
=x+\frac{a^3}{(b^2-a^2)}\arctan\frac{x}{a}-\frac{b^3}{(b^2-a^2)}\arctan\frac{x}{b}+c,
$$

where c is an integration constant.

Answer: $I=\int\frac{x^4dx}{(x^2+a^2)(x^2+b^2)}=x+\frac{a^3}{(b^2-a^2)}\arctan\frac{x}{a}-\frac{b^3}{(b^2-a^2)}\arctan\frac{x}{b}+c.$