Answer on Question #51672 – Math – Integral Calculus

I want to integrate this $\int x\sqrt{a^2-x^2}dx$, let x = a*sin(z); differentiating both sides with respect to x, it becomes

1 = a*cos(z) dz/dx or dx = a*cos(z) dz.

If I differentiate both side with respect to z, it becomes

dx/dz = a*cos(z) or dx = a*cos(z) dz.

My question is which one is correct? Differentiating with respect of x or with respect of z and why?

Solution

Both versions are correct. Your purpose is to find dx in form: dx = f(z)dz.

$$\frac{dx}{dz}=\frac{1}{\frac{dz}{dx}}.$$

So, it doesn't matter how to do differentiation: dx/dz = f(z); dz/dx = 1/f(z).

From these two cases we obtain: dx = f(z)dz.

P.S.

Method 1

I show you an easier way:

$$\int x\sqrt{a^2-x^2}dx = |\mathrm{d(x2)} = \mathrm{2xdx}| = \frac{1}{2}\int\sqrt{a^2-x^2}\,d(x^2) = \frac{a}{2}\int\sqrt{1-\frac{x^2}{a^2}}d(x^2) = -\frac{a^3}{2}\int\sqrt{1-\frac{x^2}{a^2}}d\left(-\frac{x^2}{a^2}\right) =$$

$$= -\frac{a^3}{2}\int\sqrt{1-\frac{x^2}{a^2}}d\left(1-\frac{x^2}{a^2}\right) = \left|1-\frac{x^2}{a^2}=z\right| = -\frac{a^3}{2}\int z^{\frac{1}{2}}dz = -\frac{2a^3}{3*2}z^{\frac{3}{2}}+C = -\frac{\boldsymbol{a^3}}{\mathbf{3}}\left(\mathbf{1}-\frac{\boldsymbol{x^2}}{\boldsymbol{a^2}}\right)^{\frac{3}{2}}+\boldsymbol{C} = -\frac{\mathbf{1}}{\mathbf{3}}(\boldsymbol{a^2}-\boldsymbol{x^2})^{\frac{3}{2}}+\boldsymbol{C},$$

where *C* is an arbitrary real constant.

Method 2

This method is a straightforward consequence of your question.

Assume that $\sqrt{a^2-x^2}=\sqrt{a^2-a^2sin^2z}=\sqrt{a^2cos^2z}=a|cosz|=a\cdot cosz$. Then

$\int x\sqrt{a^2 - x^2}dx = |x = a \cdot sinz, dx = a \cdot coszdz| = \int a \cdot sinz \cdot a^2 \frac{cos^2 zdz}{2} = -\frac{a^3}{2}\int cos^2 z \cdot (-sinzdz) = -\frac{a^3}{2}\int cos^2 zd(cosz) = -\frac{a^3}{2}\frac{cos^3 z}{3} + C = -\frac{a^3}{3}\left(1 - \frac{x^2}{a^2}\right)^{\frac{3}{2}} + C = -\frac{1}{3}(a^2 - x^2)^{\frac{3}{2}} + C,$

where *C* is an arbitrary real constant.