Answer on Question #51667 – Math – Multivariable Calculus

A consumer utility function given $u(x,y) = \ln\left(x + 2y - \frac{y^2}{2}\right)$. Find indirect utility function of consumer.

Solution

Because $x + 2y - \frac{y^2}{2}$ is the expression under logarithmic function, we conclude that $x + 2y - \frac{y^2}{2} > 0$.

Indirect utility function is solution to the following optimization problem:

$$V(p_1, p_2, l) = Max(u(x,y)) \qquad (1)$$

subject to $p_1 x + p_2 y \le l, x \ge 0, y \ge 0.$

The Lagrangian is $L = u(x,y) - \lambda(p_1 x + p_2 y - l) = \ln\left(x + 2y - \frac{y^2}{2}\right) - \lambda(p_1 x + p_2 y - l).$

The Kuhn-Tucker conditions are given by

$$\frac{\partial L}{\partial x} = \frac{1}{x+2y-\frac{y^2}{2}} - \lambda p_1 = 0,\ \frac{\partial L}{\partial y} = \frac{2-y}{x+2y-\frac{y^2}{2}} - \lambda p_2 = 0,\ \frac{\partial L}{\partial \lambda} = -(p_1 x + p_2 y - l) = 0 \qquad (2)$$

Consider $\frac{1}{x+2y-\frac{y^2}{2}} = \lambda p_1, \frac{2-y}{x+2y-\frac{y^2}{2}} = \lambda p_2$, solve for λ, obtain $\frac{1}{\left(x+2y-\frac{y^2}{2}\right)p_1} = \frac{2-y}{\left(x+2y-\frac{y^2}{2}\right)p_2}$, multiply both sides by $x + 2y - \frac{y^2}{2} > 0$, which gives $\frac{1}{p_1} = \frac{2-y}{p_2}$, it is equivalent to $\frac{y}{p_2} = \frac{2}{p_2} - \frac{1}{p_1}$ and finally obtain

$$y^* = y = 2 - \frac{p_2}{p_1} \qquad (3)$$

Take the last equation

$$p_1 x + p_2 y - l = 0 \qquad (4)$$

from the Kuhn-Tucker conditions (2), recall (3) and solve (4) for x:

$$x = \frac{l}{p_1} - \frac{p_2}{p_1} y = \frac{l}{p_1} - \frac{p_2}{p_1}\left(2 - \frac{p_2}{p_1}\right) = \frac{l}{p_1} - 2\frac{p_2}{p_1} + \left(\frac{p_2}{p_1}\right)^2,$$

that is,

$$x^* = x = \frac{l}{p_1} - 2\frac{p_2}{p_1} + \left(\frac{p_2}{p_1}\right)^2 \qquad (5)$$

Then plug solutions y^*, x^* from (3) and (5) back into $u(x,y)$ to get $V(p_1, p_2, l)$ in (1):

$$V(p_1, p_2, l) = Max\big(u(x,y)\big) = u(x^*, y^*) = \ln\left(x^* + 2y^* - \frac{(y^*)^2}{2}\right) =$$

$$= \ln\left(\frac{l}{p_1} - 2\frac{p_2}{p_1} + \left(\frac{p_2}{p_1}\right)^2 + 4 - 2\frac{p_2}{p_1} - \frac{1}{2}\left(2 - \frac{p_2}{p_1}\ \right)^2\right) =$$

$$= \ln\left(\frac{l}{p_1} - 2\frac{p_2}{p_1} + \left(\frac{p_2}{p_1}\right)^2 + 4 - 2\frac{p_2}{p_1} - 2 + 2\frac{p_2}{p_1} - \frac{1}{2}\left(\frac{p_2}{p_1}\ \right)^2\right) = \ln\left(\frac{l}{p_1} - 2\frac{p_2}{p_1} + \frac{1}{2}\left(\frac{p_2}{p_1}\right)^2 + 2\right).$$

Answer: $\ln\left(\frac{l}{p_1} - 2\frac{p_2}{p_1} + \frac{1}{2}\left(\frac{p_2}{p_1}\right)^2 + 2\right).$