Answer on Question #51630 – Math – Integral Calculus

Question

What is the integration of 1/[{sinx}^4 + {cosx}^4]?

Solution

Method 1

$$\int \frac{1}{\sin^4 x + \cos^4 x} dx = \int \frac{1}{\sin^4 x + \cos^4 x} dx = \int \frac{1}{\cos^4 x\left(\frac{\sin^4 x}{\cos^4 x} + 1\right)} dx = \int \frac{1}{\cos^2 x(\tan^4 x + 1)} \frac{dx}{\cos^2 x} =$$

$$= \int \frac{1}{\cos^2 x(\tan^4 x + 1)} \frac{dx}{\cos^2 x} = \int (\tan^2 x + 1) \frac{1}{\tan^4 x + 1} d\tan x = |t = \tan x| = \int \frac{(t^2 + 1)}{t^4 + 1} dt =$$

$$= \int \frac{(t^2 + 1)}{t^4 + 1} dt = \int \frac{t^2 + 1}{(t^4 + 2t^2 + 1 - 2t^2)} dt = \int \frac{t^2 + 1}{(t^2 + 1 - t\sqrt{2})(t^2 + 1 + t\sqrt{2})} dt$$

$$\frac{t^2 + 1}{(t^2 - t\sqrt{2} + 1)(t^2 + t\sqrt{2} + 1)} = \frac{At + B}{t^2 - t\sqrt{2} + 1} + \frac{Ct + D}{t^2 + t\sqrt{2} + 1} = \frac{At^3 + Bt^2 + At^2\sqrt{2} + Bt\sqrt{2} + At + B}{(t^2 - t\sqrt{2} + 1)(t^2 + t\sqrt{2} + 1)} +$$

$$+ \frac{Ct^3 + Dt^2 - Ct^2\sqrt{2} - Dt\sqrt{2} + Ct + D}{(t^2 - t\sqrt{2} + 1)(t^2 + t\sqrt{2} + 1)} =$$

$$= \frac{(A + C)t^3 + (B + D)t^2 + (A - C)t^2\sqrt{2} + (B - D)t\sqrt{2} + (A + C)t + B + D}{(t^2 - t\sqrt{2} + 1)(t^2 + t\sqrt{2} + 1)},$$

$A + C = 0$, $B + D = 1$, $A - C = 0$, $B - D = 0$, hence

$\begin{cases} A + C = 0 \\ A - C = 0 \end{cases}$, which gives $\begin{cases} 2A = 0 \\ A = C \end{cases}$, so $A = C = 0$

$\begin{cases} B + D = 1 \\ B = D \end{cases}$, which gives $2B = 1$, $B = D$, so $B = D = \frac{1}{2}$

obtain

$$\frac{t^2 + 1}{(t^2 - t\sqrt{2} + 1)(t^2 + t\sqrt{2} + 1)} = \frac{1}{2(t^2 - t\sqrt{2} + 1)} + \frac{1}{2(t^2 + t\sqrt{2} + 1)} =$$

$$= \frac{1}{2\left(\left(t - \frac{\sqrt{2}}{2}\right)^2 + \frac{1}{2}\right)} + \frac{1}{2\left(\left(t + \frac{\sqrt{2}}{2}\right)^2 + \frac{1}{2}\right)}$$

$$\int \frac{t^2+1}{\left(t^2+1-t\sqrt{2}\right)\left(t^2+1+t\sqrt{2}\right)}dt = \frac{1}{2}\int \frac{dt}{\left(\left(t-\frac{\sqrt{2}}{2}\right)^2+\frac{1}{2}\right)} + \frac{1}{2}\int \frac{dt}{\left(\left(t+\frac{\sqrt{2}}{2}\right)^2+\frac{1}{2}\right)} =$$

$$= \frac{\sqrt{2}}{2}\arctan\frac{t-\frac{\sqrt{2}}{2}}{\frac{1}{\sqrt{2}}} + \frac{\sqrt{2}}{2}\arctan\frac{t+\frac{\sqrt{2}}{2}}{\frac{1}{\sqrt{2}}} + c = \frac{\sqrt{2}}{2}\arctan\left(t\sqrt{2}-1\right) + \frac{\sqrt{2}}{2}\arctan\left(t\sqrt{2}+1\right) + c =$$

$$= \frac{\sqrt{2}}{2}\arctan\left(\sqrt{2}\tan x - 1\right) + \frac{\sqrt{2}}{2}\arctan\left(\sqrt{2}\tan x + 1\right) + c,$$

where arctan is the inverse of tan function, *c* is an arbitrary real constant.

Method 2

$$\sin^4 x + \cos^4 x = (\sin^2 x + \cos^2 x)^2 - 2\sin^2 x\cos^2 x = 1 - 2\sin^2 x\cos^2 x = 1 - \frac{1}{2}\sin^2 2x =$$

$$= (1 + \frac{\sqrt{2}}{2}\sin 2x)(1 - \frac{\sqrt{2}}{2}\sin 2x)$$

$$\int \frac{1}{\sin^4 x + \cos^4 x}dx = \int \frac{1}{(1+\frac{\sqrt{2}}{2}\sin 2x)(1-\frac{\sqrt{2}}{2}\sin 2x)} = \frac{1}{2}[\int \frac{1}{1+\frac{\sqrt{2}}{2}\sin 2x}dx + \int \frac{1}{1-\frac{\sqrt{2}}{2}\sin 2x}dx]$$

$$\sin 2x = 2\sin x\cos x = \frac{2\sin x}{\cos x}\cos^2 x = \frac{2\tan x}{\frac{1}{\cos^2 x}} = \frac{2\tan x}{1+\tan^2 x} = [t = \tan x] = \frac{2t}{1+t^2}$$

$$dt = d(\tan x) = \frac{1}{\cos^2 x}dx = (1+t^2)dx \Rightarrow dx = \frac{dt}{1+t^2}$$

$$\frac{1}{2}[\int \frac{1}{1+\frac{\sqrt{2}}{2}\sin 2x}dx + \int \frac{1}{1-\frac{\sqrt{2}}{2}\sin 2x}dx] = \frac{1}{2}[\int \frac{\frac{dt}{1+t^2}}{1+\frac{\sqrt{2}}{2}(\frac{2t}{1+t^2})} + \int \frac{\frac{dt}{1+t^2}}{1-\frac{\sqrt{2}}{2}(\frac{2t}{1+t^2})}] =$$

$$= \frac{1}{2}[\int \frac{dt}{1+t^2+\sqrt{2}t} + \int \frac{dt}{1+t^2-\sqrt{2}t}] = \frac{1}{2}[\int \frac{dt}{(t+\frac{1}{\sqrt{2}})^2+\frac{1}{2}} + \int \frac{dt}{(t-\frac{1}{\sqrt{2}})^2+\frac{1}{2}}] =$$

$$= \frac{1}{2}[\int \frac{2dt}{(\sqrt{2}t+1)^2+1} + \int \frac{2dt}{(\sqrt{2}t-1)^2+1}] = \frac{1}{\sqrt{2}}(\arctan(\sqrt{2}\tan x + 1) + \arctan(\sqrt{2}\tan x - 1))$$

So $\int \frac{1}{\sin^4 x + \cos^4 x}dx = \frac{1}{\sqrt{2}}(\arctan(\sqrt{2}\tan x + 1) + \arctan(\sqrt{2}\tan x - 1)) + c$,

where arctan is the inverse of tan function, c is an arbitrary real constant.