Answer on Question #51516 – Math – Differential Geometry

Question

1) Find the point of maximum curvature on the graph of y = e^x

2) Find the minimum and maximum curvatures of ellipse r(t) = (a*cos t, b*sin t), 0<=t<=2pi, where a>b.

Solution

1) Curvature is given by

$$k(x) = \frac{y''}{(1+(y')^2)^{3/2}} = \frac{e^x}{(1+e^{2x})^{3/2}}.$$

To find the point of maximum, we can start with the first derivative of function k(x):

$$k'(x) = \frac{e^x(1+e^{2x})^{\frac{3}{2}} - \left(\frac{3}{2}\right)(1+e^{2x})^{\frac{1}{2}} * 2 * e^{2x}e^x}{(1+e^{2x})^3} = 0;$$

$$k'(x) = \frac{e^x(1+e^{2x})^{\frac{1}{2}}}{(1+e^{2x})^3}(1+e^{2x}-3e^{2x}) = 0;$$

$$k'(x) = \frac{e^x(1+e^{2x})^{\frac{1}{2}}}{(1+e^{2x})^3}(1-2e^{2x}) = 0;$$

$$e^{2x} = \frac{1}{2};$$

$$x_0 = \frac{1}{2}ln\frac{1}{2}.$$

k'(x) is going to be negative for x greater than x_0 and positive for x < x_0. By the first sufficient condition of maximum, x_0 is the maximum value of curvature k(x).

Answer: $x_0 = \frac{1}{2}ln\frac{1}{2}$.

2) x(t) = acos t;   y(t) = bsin t, $0 \le t \le 2\pi$.

Curvature is given by

$$k(t) = \frac{x'y''-y'x''}{(x'^2+y'^2)^{\frac{3}{2}}} = \frac{ab\,sin(t)\sin(t)+abcos(t)\cos(t)}{\left((-\operatorname{asin}(t))^2+(bcos(t))^2\right)^{\frac{3}{2}}} = \frac{ab\,(\sin^2(t)+\cos^2(t))}{\left((\operatorname{asin}(t))^2+(bcos(t))^2\right)^{\frac{3}{2}}} =$$

$$= \frac{ab}{((\operatorname{asin}(t))^2+(bcos(t))^2)^{3/2}} = \frac{ab}{(b^2+(a^2-b^2)sin^2(t))^{3/2}}.$$

This quantity is minimum when denominator is maximum, i.e. for $\sin^2 t = 1$ where $a>b$. The minimum curvature points are at **$t = t_{min} = \pi/2; 3\pi/2$,** hence **$x = 0; y = \pm b$** and the minimum curvature is **$k_{min} = \frac{ab}{a^3} = \frac{b}{a^2}$,** where $a>b$.

This quantity is maximum when denominator is minimum, i.e. for $\sin^2 t = 0$ where $a>b$. The maximum curvature points are at **$t = t_{max} = 0; \pi$,** hence **$x = \pm a; y = 0$** and the maximum curvature is **$k_{max} = \frac{ab}{b^3} = \frac{a}{b^2}$,** where $a>b$.

Note that the maximum curvature is $k_{max} = \frac{b}{a^2}$ and the minimum curvature is $k_{min} = \frac{a}{b^2}$, where $a<b$.

Answer: **$k_{min} = b/a^2$; $k_{max} = a/b^2$, where $a>b$.**